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Pressure of a prestressed cylinder on an elastic layer which lies without friction on an elastic basis with initial stresses

Abstract. *The article is devoted to the research of contact interaction of prestressed bodies, namely: the pressure of an elastic cylinder die upon an elastic layer and foundation with initial (residual) stresses. The following case is considered in the article: the elastic cylinder with initial stresses presses on a prestressed layer, which lies without friction on the basis with initial stresses. The problem is solved for the case of equal roots of the resolving equation. And the study is presented in a general form for the theory of large initial deformations and two versions of the theory of small initial deformations in the framework of the linearized theory of elasticity for an arbitrary structure of the elastic potential.*

Key words: *the linearized elasticity theory, initial (residual) stresses, contact problem, punch, cylinder die.*

Introduction. Applied needs of science, modern technology and new technologies created the need to predict contact behavior of various structures, and in recent decades stimulated the development of various mathematical models and methods of contact mechanics of bodies with different properties.

One of the important factors influencing contact interactions is the effect of the initial stress, which is almost always present in real structures and parts of machines, thus making development of effective methods of calculation of stress-strain state, taking into account the initial deformation, relevant and important scientific and technical challenge.

Today results on wide range of issues concerning contact problems for elastic, viscoelastic and plastic bodies have been obtained. These results are sufficiently presented in numerous periodical publications. Despite the significant achievements the number of research on contact interaction of bodies with initial stresses is relatively small. Detailed review of problems of contact interaction of elastic bodies with initial stresses is presented in [1].

The first works on the contact interaction of bodies with initial stresses describe interaction of pre-stressed rigid bodies with rigid and elastic punches without initial stress [1]. Thus either elastic potentials of concrete structures are considered, or the task relates in general to compressible (incompressible) bodies with the potential of arbitrary structure on the basis of linearized elasticity theory. There is also a number of other generalizing publications that are partially related to the topic of this article [2, 3].

This paper, using ratios of linearized elasticity theory [1] presents a solution to the contact problem of elastic cylindrical punch with initial stresses pressing on the elastic layer and the base with initial stresses. The following case is considered: layer with initial stress is positioned without friction on a base with initial stress. Research is done in general for compressible (incompressible) bodies for the theory of large initial deformations and two versions of small initial deformations theory with the random structure of elastic potential. We believe that initial deflected modes in the layer, punch and the base are homogeneous and equal, and elastic potentials are twice continuously-differentiable functions of Green strain tensor algebraic invariants [1].

For research purposes Lagrange coordinates (x_1, x_2, x_3) coinciding in their initial state with Cartesian coordinates (y_1, y_2, y_3), with related ratios $y_i = \lambda_i x_i \quad (i = \overline{1,3})$ have been introduced, where λ_i – elongation factors that determine the movement of the original (remaining) state; y_i – coordinates of the initial strain state; x_i – Lagrange coordinates. Punch, layer and base materials are considered isotropic compressible or incompressible. In the case of orthotropic materials it is assumed that elastic-equivalent directions coincide with the directions of the coordinate axes.

All values relating to the elastic cylindrical punch are marked with superscript "(1)", those referring to the layer and the foundation are denoted with upper indexes "(2)" and "(3)" respectively.

Statement of the problem. Consider elastic cylindrical punch (Fig. 1) with radius *R* and height *H* with the initial stress that is pressed into the elastic layer under the influence of force *P* after the initial strain state occurred. The thickness of the layer in its original deformed state is related to the thickness of its unstrained state by $h_1 = \lambda_3 h_2$ relation. We assume that the external load is applied only to the free end of the elastic punch, under which all points of punch move towards the axis of symmetry y_3 by the same value ε. We assume that the surface outside the contact area remains free from the influence of external forces, and no friction occurs in the contact zone while the movement and stress are continuous.

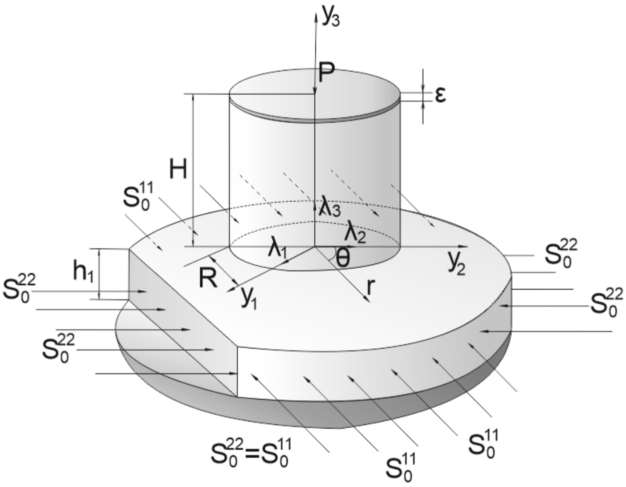


Fig. 1. Pressure of a cylinder on a layer and a foundation with initial stresses.

Given the condition of the existence of a single solution to linearized elasticity theory for compressible (incompressible) bodies [1], there are two options of presenting the total solution [1]: in case of equal roots $(\xi_2'^2 = \xi_3'^2)$ and a case of unequal roots $(\xi_2'^2 \neq \xi_3'^2)$ [2]. In this article we are considering the case of equal roots:

$$\tilde{\chi} = \tilde{\chi}_1 + y_3\tilde{\chi}_2, \left(\Delta_1 + \xi_2'^2\, \partial^2/\partial y_3^2\right)\tilde{\chi}_1 = 0, \ \left(\Delta_1 + \xi_2'^2\, \partial^2/\partial y_3^2\right)\tilde{\chi}_2 = 0 \qquad (1)$$

In the circular cylindrical coordinate system (r, θ, z_i), where $z_i = \nu_i^{-1} y_3$, $\nu_i = \sqrt{n_i}$, $(i = \overline{1,\ 2})$, $n_1 = \xi_2'^2$, $n_2 = \xi_3'^2$, this formulation corresponds to the following boundary conditions: at the end of an elastic punch:

$$u_3^{(1)} = -\varepsilon; \quad \tilde{Q}_{3r}^{(1)} = 0 \quad (0 \le r \le R) \qquad (2)$$

on the elastic layer boundary in the contact area:

$$u_3^{(1)} = u_3^{(2)}; \quad \tilde{Q}_{33}^{(1)} = \tilde{Q}_{33}^{(2)} \quad \tilde{Q}_{3r}^{(1)} = \tilde{Q}_{3r}^{(2)} = 0 \quad (0 \le r \le R) \qquad (3)$$

on the elastic layer boundary outside the contact area:

$$\tilde{Q}_{33}^{(2)} = 0 \quad \tilde{Q}_{3r}^{(2)} = 0 \quad (R \le r < \infty); \qquad (4)$$

on the side surface of the elastic punch $r = R$:

$$\tilde{Q}_{rr}^{(1)} = 0; \quad \tilde{Q}_{3r}^{(1)} = 0 \quad (0 \le z_i \le \frac{H}{v_i}). \tag{5}$$

At the bottom surface of the layer that lies without friction on the elastic basis with initial stresses $z_i = -\frac{\lambda_3 h_2}{v_i} = -\frac{h_i}{v_i}$, $\left(i = \overline{1,2}\right)$:

$$u_3'^{(2)} = u_3'^{(3)}, \quad Q_{3r}'^{(3)} = Q_{3r}'^{(2)} = 0, \quad (0 \le r < \infty), \tag{6}$$

where $z_i = y_3/v_i$ (i=1,2) the thickness of the layer in the unstrained state.

The equilibrium condition, which establishes the connection between the end subsidence and resultant load P has the form

$$P = -2\pi R^2 \int_0^1 \rho Q_{33}^{(2)}(0,\rho) d\rho. \tag{7}$$

To determine the stress-strain state of elastic cylinder in case of equal roots general solution of defining equation (1) takes the form:

$$\hat{\chi} = \varepsilon \Big\langle v_1 z_1 (1+z_1) \Big[(m_2 - 1)^{-1} + \chi_0 \Big((1-m_2)^{-1} - 2E(3H\theta_2)^{-1}(3r^2 - 2z_1^2) \Big) \Big] + \tag{8}$$

$$+R\sum_{k=1}^{\infty} \chi_k \Big[R(2\gamma_k)^{-1} b_1^{(k)} \Big(H\Big(1 + \frac{s_0(1 - I_0(v_1\gamma_k R))}{v_1 \gamma_k R I_1(v_1\gamma_k R)}\Big) + z_1 \Big) I_0(\gamma_k v_1 r) S_1(\gamma_k z_1 v_1) + J_0(\alpha_k r)\mu_k^{-1}\Big(\tilde{S}_2(\alpha_k z_1) + z_1 \tilde{S}_3(\alpha_k z_1)\Big)\Big]\Big\rangle$$

where $s_0 = (1+m_2)/(1+m_1)^{-1}$, $\tilde{S}_2(\alpha_k z_1) = Rs_0\mu_k^{-1} ch(\mu_k z_1 R^{-1}) + E^{(k)} sh(\mu_k z_1 R^{-1})$, $S_1 = \sin(\gamma_k v_1 z_1)$, $b_1^{(k)}$ – is expressed by the boundary conditions (2)–(6), $\tilde{S}_3(\alpha_k z_2) = -sh(\mu_k z_1 R^{-1}) - M^{(k)} ch(\mu_k z_1 R^{-1})$, $\theta_2 = E\left(8m_1(1+H)n_1^{-1} - 4Hv_1^{-1} + (1-m_2)R^2 H^{-1}\right)$.

Then expressions for the components of the movement vector and strain tensor for a cylindrical punch will look as:

$$U_3'^{(1)} = \varepsilon \Big\langle (m_2 - 1)v_1^{-1} + \Big[1 - 2E(H\theta_2)^{-1}\Big(r^2 - 2z_1^2 + 4m_1 z_1 \left(v_1^{-1} + z_1\right)\Big)\Big]\chi_0 +$$

$$+\sum_{k=1}^{\infty} \chi_k \Big\{ 0,5R^2 b_1^{(k)} \gamma_k I_0(\gamma_k v_1 r) \Big[\Big(H\Big(1 + s_0(1 - I_0(v_1\gamma_k R))(v_1\gamma_k R I_1(v_1\gamma_k R))^{-1}\Big) + v_1 z_1 \Big) m_1 \sin(\gamma_k v_1 z_1) +$$

$$+(1-m_2)\cos(\gamma_k v_1 z_1)\gamma_k^{-1}\Big] - J_0(\alpha_k r) n_1^{-1} \Big[m_1 \alpha_k \left(\tilde{S}_2(\alpha_k z_1) + z_1 v_1 \tilde{S}_3(\alpha_k z_1)\right) + (m_2 - 1) v_1 \tilde{S}_5(\alpha_k z_1)\Big]\Big\}\Big\rangle,$$

$$Q_{33}'^{(1)} = C_{44}\varepsilon \Big\langle -8Ev_1(H\theta_2 R^2)^{-1}\chi_0 \Big[(1+m_1) l_1 \left(v_1^{-1} + z_1\right) + (1+m_2) l_2 z_1 \Big] + \tag{9}$$

$$+\sum_{k=1}^{\infty} \chi_k \Big\{ 0,5R^2 b_1^{(k)} \gamma_k n_1 I_0(\gamma_k v_1 r) \Big[(1+m_1) l_1 \gamma_k \Big(H\Big(1 + s_0(1 - I_0(v_1\gamma_k R))(v_1\gamma_k R I_1(v_1\gamma_k R))^{-1}\Big) + v_1 z_1 \Big) \cos(\gamma_k v_1 z_1) +$$

$$+(1+m_2) l_2 \sin(\gamma_k v_1 z_1)\Big] - \alpha_k J_0(\alpha_k r) \Big[(1+m_1) l_1 \alpha_k v_1^{-1} \left(\tilde{S}_4(\alpha_k z_1) + v_1 z_1 \tilde{S}_5(\alpha_k z_1)\right) + (1+m_2) l_2 \tilde{S}_3(\alpha_k z_1)\Big]\Big\}\Big\rangle$$

where $\tilde{S}_4(\alpha_k z_1) = Rs_0\mu_k^{-1}sh(\mu_k z_1 R^{-1}) + E^{(k)}ch(\mu_k z_1 R^{-1}),\ \ \tilde{S}_5(\alpha_k z_2) = -ch(\mu_k z_1 R^{-1}) - M^{(k)}sh(\mu_k z_1 R^{-1}),$

Stressed-deformed state of elastic layer with initial stresses is determined by harmonic functions in the form of Hankel radial integrals. Having satisfied condition (2) – (6), after a number of changes we will have:

$$Q_{33}'^{(2)} = (1+m_1)\varepsilon l_1 C_{44}(\pi\theta_3 R)^{-1}\tilde{T}^1(\Omega_+^5; S_2^0; N_1^0; K_1^0; s),\ \ u_3'^{(2)} = m_1\varepsilon(\pi\theta_3 v_1)^{-1}\tilde{T}^1(\Omega_-^5; S_1^0; N_0^0; K_0^0; s_1) \quad (10)$$

$$Q_{33}'^{(3)} = 2\varepsilon(\mu+\lambda)(\pi R\theta_3)^{-1}A_0'\tilde{T}^1\left(\Omega_-^7; S_3^0; N_2^0; K_2^0; 1; 1\right), u_3'^{(3)} = \varepsilon(\pi\theta_3)^{-1}A_0'\tilde{T}^1\left(\Omega_-^7; S_2^0; N_1^0; K_1^0; (2\mu-\lambda)\mu^{-1}; (\lambda+\mu)\mu^{-1}\right)$$

where $\tilde{T}^1(\Omega_\pm^l; S_{m_1}^n; N_{m_2}^n; K_{m_3}^n; \mathrm{k}; \mathrm{a}) = (1+a_0)\left\langle (1-\chi_0)\Omega_\pm^l\left(S_{m_1}^n; 0; \mathrm{k}; \mathrm{a}; 0\right) - \frac{\theta_3}{\varepsilon}\sum_{j=0}^{\infty} C_j^{**}\Omega_\pm^l\left(S_{j+m_1}^n; 0; \mathrm{k}; \mathrm{a}; 0\right) - \right.$

$$\left. -\frac{2(m_2-1)R^2}{\theta_2}\chi_0\Omega_\pm^l\left(N_{m_2}^n; 0; \mathrm{k}; \mathrm{a}; 0\right) + \theta_4\sum_{k=1}^{\infty}\chi_k\Omega_\pm^l\left(K_{m_3}^n; \mu_k; \mathrm{k}; \mathrm{a}; 0\right) + \frac{(m_2-1)R^2}{2}\sum_{k=1}^{\infty} b_1^{(k)}\chi_k\Omega_\pm^l\left(K_{m_3}^n; i\gamma_k v_1 R; \mathrm{k}; \mathrm{a}; 0\right)\right\rangle +$$

$$+\sum_{\tau=1}^{\infty} a_\tau\left\langle (1-\chi_0)\Omega_\pm^l\left(S_{m_1}^n; 0; \mathrm{k}; \mathrm{a}; v_1\tau\right) - \frac{\theta_3}{\varepsilon}\sum_{j=0}^{\infty} C_j^{**}\Omega_\pm^l\left(S_{j+m_1}^n; 0; \mathrm{k}; \mathrm{a}; v_1\tau\right) - \frac{2(m_2-1)R^2}{\theta_2}\chi_0\Omega_\pm^l\left(N_{m_2}^n; 0; \mathrm{k}; \mathrm{a}; v_1\tau\right) + \right.$$

$$\left. +\theta_4\sum_{k=1}^{\infty}\chi_k\Omega_\pm^l\left(K_{m_3}^n; \mu_k; \mathrm{k}; \mathrm{a}; v_1\tau\right) + \frac{(m_2-1)R^2}{2}\sum_{k=1}^{\infty} b_1^{(k)}\chi_k\Omega_\pm^l\left(K_{m_3}^n; i\gamma_k v_1 R; \mathrm{k}; \mathrm{a}; v_1\tau\right)\right\rangle$$

$$\Omega_\pm^5(\hat{\mathrm{L}}_m^n, \mu, \mathrm{k}, \theta) = (A_1^{03} + kA_2^{03} - s_3 + \mathrm{k})\hat{\mathrm{L}}_m^n\left(\rho, \mu, z_1R^{-1} - \theta\right) + \left(A_1^{13} + kA_2^{13} + z_1R^{-1}(A_2^{03}+1)\right)\hat{\mathrm{L}}_{m+1}^n\left(\rho, \mu, z_1R^{-1} - \theta\right) +$$

$$+z_1R^{-1}A_2^{13}\left(\hat{\mathrm{L}}_{m+2}^n\left(\rho, \mu, z_1R^{-1} - \theta\right) \mp \hat{\mathrm{L}}_{m+2}^n\left(\rho, \mu, -z_1R^{-1} - \theta\right)\right) \pm (A_1^{03} + kA_2^{03} + s_3 - \mathrm{k})\hat{\mathrm{L}}_m^n\left(\rho, \mu, -z_1R^{-1} - \theta\right) \pm$$

$$\pm\left(A_1^{13} + kA_2^{13} - z_1R^{-1}(A_2^{03}-1)\right)\hat{\mathrm{L}}_{m+1}^n\left(\rho, \mu, -z_1R^{-1} - \theta\right),\ \ N_n^m(\rho; z) = \int_0^\infty \eta^n\psi_1(\eta, 0)e^{z\cdot\eta}J_m(\eta\rho)d\eta,$$

$$\Omega_\pm^7(\hat{\mathrm{L}}_m^n; \mu; \mathrm{k}; \mathrm{a}; \theta) = \mathrm{k}\cdot\hat{\mathrm{L}}_m^n\left(\rho, \mu, h_1(Rv_1)^{-1} + y_3R^{-1} - \theta\right) \pm \mathrm{a}(h_1 + y_3)R^{-1}\hat{\mathrm{L}}_{m+1}^n\left(\rho, \mu, h_1(Rv_1)^{-1} + y_3R^{-1} - \theta\right),$$

$$\hat{\mathrm{L}}_m^n(\mathrm{t}, 0, \mathrm{u}) = \hat{\mathrm{L}}_m^n(\mathrm{t}, \mathrm{u}),\ \ S_n^m(\rho; z) = \int_0^\infty \eta^{n-2}\sin\eta e^{-z\cdot\eta}J_m(\eta\rho)d\eta,\ \ K_n^m(\rho; \mu_k; z) = \int_0^\infty \eta^n\psi_0(\eta, \mu_k)e^{z\cdot\eta}J_m(\eta\rho)d\eta,$$

k_i, a_i, A_0' – certain constants (i=0,1,2,...), coefficients $n_i,\ m_i,\ c_{44},\ l_i$ are given in [1].

The method of solution. Using the solutions for cylinder (8), (9) and satisfying the condition (3) – (5), we find the eigenvalues of the problem (2) - (7) in the case of equal roots: $\gamma_k = 2\pi kH^{-1},\ \ (k = 0,1,2,...)$, $\alpha_k = \frac{\mu_k}{R}$, where $J_1(\mu_k) = 0$.

With the first conditions (3) and (4) we can determine the unknown function $F(\eta)$ of dual integral equations for equal roots:

$$\int_0^\infty F(\eta)\eta^{-1}J_0(\eta\rho)d\eta = f(\rho),\ (\rho < 1),\quad \int_0^\infty F(\eta)J_0(\eta\rho)d\eta = 0,\ (\rho > 1), \quad (11)$$

where $f(\rho) = \theta_3^{-1}\left(-(m_2-1)v_1^{-1}(A_0\psi_0(\eta,0) + 3C_0R^2\psi_1(\eta,0)) - \mu_k(n_1R)^{-1}(m_1\mu_kR^{-1}F_k + v_1(m_2-1)N_k)\psi_0(\eta,\mu_k) + \right.$

$$+\gamma_k(1-m_2)B_k\psi_0(\eta,i\gamma_k v_1 R)\big)+\int_0^\infty \frac{F(x)G(xh)\psi_0(\eta x)}{x}dx\,,\quad \theta_3=\frac{m_1}{v_1}(s_1-s_0),$$

Applying reciprocation formula to (11) leads to a Fredholm integral equation of the second kind regarding function $F(\eta)$:

$$\frac{F(\eta)}{\eta}=-\frac{2\varepsilon}{\pi\theta_3}\Bigg((1-\chi_0)\psi_0(\eta,0)-2(m_2-1)\frac{R^2}{\theta_2}\chi_0\psi_1(\eta,0)+\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+$$
$$+0,5(m_2-1)R^2\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\psi_0(\eta,i\gamma_k v_1 R)\Bigg)+2\pi^{-1}\int_0^\infty u^{-1}F(u)G(uh)\psi_0(\eta,u)du \qquad (12)$$

where $\psi_n(x,y)=\int_0^1 t^n\cos xt\cos yt dt$.

Satisfying the second boundary condition (3), we will look for solution (12) using a method of successive approximations:

$$F(\eta)=\sum_{n=0}^{\infty}F^{(n)}(\eta)\,, \qquad (13)$$

where $F^{(0)}(\eta)\eta^{-1}=-2(\pi\theta_3)^{-1}\varepsilon\Big((1-\chi_0)\psi_0(\eta,0)-2(m_2-1)R^2\theta_2^{-1}\chi_0\psi_1(\eta,0)+\theta_4\sum_{k=1}^{\infty}\chi_k\psi_0(\eta,\mu_k)+$

$+0,5(m_2-1)R^2\sum_{k=1}^{\infty}b_1^{(k)}\chi_k\psi_0(\eta,i\gamma_k v_1 R)\Big)$, $F^{(k)}(\eta)\eta^{-1}=2\pi^{-1}\int_0^\infty u^{-1}F^{(k-1)}(u)G(uh)\psi_0(\eta,u)du$.

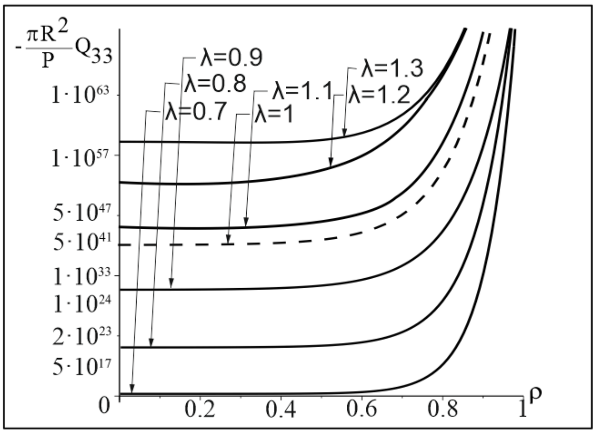
Note that the process of successive approximations (13) is convergent at $h>1$, but due to the bulkiness its proof is not presented here. Satisfying the boundary conditions (3) given orthogonality of Bessel functions $J_0(\mu_k\rho)$ to determine constants χ_i $(i=0,1,2,..)$ we obtain an infinite system of algebraic equations:

$$\vartheta_k\chi_k+\sum_{n=0}^{\infty}\vartheta_{kn}\chi_n=\varpi_k \quad (k=0,1,2,..)\,. \qquad (14)$$

On defining the unknown constants χ_i $(i=0,1,2,..)$ out of the system (14), we can calculate the deflected mode in both the elastic punch and the layer using the formulas (9) – (10). As a result, the solution is presented as a series through infinite system of constants determined from a system of linear equations.

The paper also provides numerical solution of (14) for harmonious potential at these parameter values: k=n=32; l=10; λ_1 =0.7;0.8;0.9;1;1.1;1.2; E=3.92. The algorithm is based on the method of reduction and is implemented as a Maple 15.

Fig. 2 and 3 show the distribution of contact stresses and waves under the punch, where values λ_1 correspond to the line, starting from the bottom to the top.



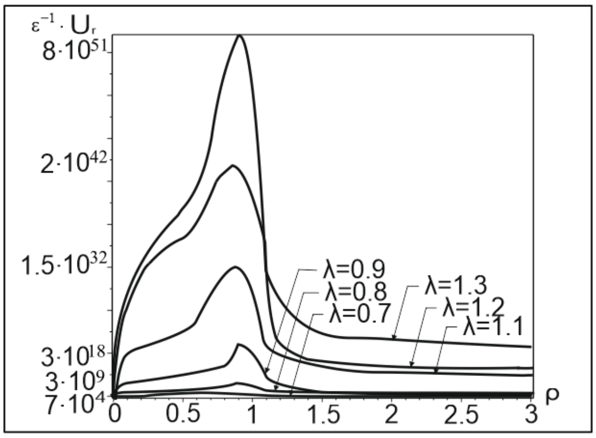


Fig. 2. Contact stresses at h=1.6

Fig. 3. Contact waves at h=1.6

In figures 2, 3 dotted lines describe the case without initial stress ($\lambda_1 = 1$), and solid lines describe the case with initial (residual) stresses.

Conclusions. Effect of initial stress on the deflected mode of elastic cylinder which is pressed into the elastic layer and the foundation is as follows: in the case of compression the original strains in a layer lead to reduction of stress in an elastic punch, whereas in case of tension they lead to their increase, and in case of movement the effects are opposite.

That is, the presence of pre-stressed state during contact interaction of elastic bodies makes it possible to adjust the contact stress and movements at structure durability calculations. Thus for contact stresses initial tensions are dangerous in case of stretching, and for movements initial stresses are dangerous in case of compression.

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