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CONTACT PROBLEM FOR AN ELASTIC RING PUNCH AND A HALF-SPACE WITH INITIAL (RESIDUAL) STRESSES*

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The problem of contact interaction without friction between an elastic cylindrical ring punch and an elastic half-space with initial (residual) stresses under pressure in the case of unequal roots of the characteristic equation is considered. The research is presented in general form for the theory of large initial (final) strains and two variants of the theory of small initial strains within the framework of the linearized theory of elasticity for an arbitrary elastic potential. Numerical analysis is presented in the form of graphs for the Treloar potential.

Keywords: linearized theory of elasticity, initial (residual) stresses, contact problem, ring punch, half-space

Introduction. Improving the reliability and durability of engineering structures and machines is one of the most urgent tasks of modern construction and mechanical engineering. The solution is greatly facilitated by scientific research within solid mechanics (especially when studying the problem of load transfer in structures and parts of machines). The emergence of new materials, the need to improve the performance of buildings and machines, reduce their weight, increase service life, reduce cost and achieve economic compatibility require new methods.

Problems related to the contact of elastic, viscoelastic, and plastic bodies without initial stresses have been studied on a number of issues. All of them are detailed in many monographs and educational works [5], as well as in publications in scientific periodicals.

The number of publications on the mechanics of contact interaction is constantly increasing, due to the urgency of the problems considered in engineering practice. But the modern needs of engineering practice have posed a number of problems that require the use of more complex models of continuous media (other than classical ones) with complex physical and mechanical properties. These models should take into account, for example, the following factors in contact interaction: friction, heat release, surface properties, surface hardness, and wear resistance, which is related to the micromechanics of frictional interaction.

One of the important factors of contact interaction (along with the others) is to take into account the initial (residual) stresses. Despite the achievements in the development of contact problems, the issue of initial stresses in contact interaction still remains insufficiently studied. Almost all structural members have initial stresses caused by various factors such as technological operations, production processes (in the manufacture of a number of materials), or assembly of structures. In the Earth's crust, the initial stresses occur due to the geostatic and geodynamic forces; in composite materials, they occur as a result of manufacturing processes. Initial stresses are also present in the blood vessels of living organisms. The initial stresses must be

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taken into account when solving problems of deformation of soils (especially frozen ones). In addition, in elastic and plastic bodies there may also be internal residual stresses after removal of the loading. Sometimes, it is appropriate to create initial stresses (residual and process-induced) intentionally to compensate for those stresses that occur in structural members during operation and increase their strength.

Due to the commercial use of new artificial materials that can withstand large initial strains, the study of contact problems for prestressed bodies is of great interest.

Thus, the mechanics of materials and structural members, geophysics, seismology, rock mechanics, composite mechanics, biomechanics, non-destructive methods for determining stresses, and others are a list of scientific areas of fundamental and applied nature, related to the need to study the effect of initial (residual) stresses or strains. It is important to study the effect of initial stresses on the stress–strain state at the contact boundary.

Taking into account the initial stresses in the design of the structural members, machines, and structures will allow more effective account for the strength resources of materials by properly assessing strength reserves and significantly reduce their material consumption while maintaining the necessary fundamental properties. Quite often, in order to increase the strength of a structure, there is a need to strengthen some of its load-bearing elements with elastic fasteners (stringers). A relevant research on a prestressed structure was performed in [13]. All the bodies in this article are elastic and prestressed.

It should be noted that so far there have been two approaches to the study of problems. The first approach involves the study of bodies with a specific elastic potential. Apparently, the first work was [20], which considers the problem for a circular crack in a prestressed elastic incompressible body for the Treloar potential (neo-Hookean body). This approach was used by V. M. Aleksandrov and N. Kh. Arutyunyan [1] and their students: S. R. Brudnyi, V. S. Poroshin, V. B. Sobol', L. M. Filipova [11], V. V. Kalinchuk, I. V. Poliakova, I. V. Anan'eva, I. V. Vorotintseva, B. I Smetanin, M. I. Chebakov, and others, as well as by R. S. Dhaliwal, J. G. Rokne, B. M. Singh [14], S. Rajit.

The second approach, which is developed in parallel with the first one, belongs to O. M. Guz [6–10] and is associated with the study of problems for prestressed elastic bodies with an arbitrary elastic potential. Problems were solved in general form for compressible and incompressible materials for the theory of large (finite) initial strains and different variants of the theory of small initial strains separately for equal and unequal roots of the characteristic (constitutive) equation [9]. All the results presented in this article were obtained with the second approach, which, we believe, has a number of advantages over the first one.

Thus, until recently, the same problem (contact or crack problem) for prestressed bodies was considered by some authors, for example, for the Treloar potential, and by other authors for the Mooney potential, etc.; i.e., for a specific form of elastic potential. In this article (as in a number of others) the study is conducted in a general form for compressible and incompressible prestressed bodies with an arbitrary elastic potential. And only at the final stage of research, specific elastic potentials are used to obtain numerical results.

All studies of contact problems for rigid and elastic punches within the second approach have been obtained in the works of academician O. M. Guz and his students: V. B. Rudnytskyi, P. P. Grigorenko, A. O. Ramsky, Yu. P. Glukhov, M. M. Dikhtiaruk, O. V. Primachenko, S. V. Matnyak, in particular the authors of this article. Research on contact problems of the authors (Ukrainian scientists) is reflected in many monographs and educational works and included in numerous publications, including review articles in Ukrainian and international periodicals. Noteworthy are [6–10, 12, 13, 15–19, 22–24].

Here we use the linearized theory of elasticity to solve the axisymmetric contact problem of pressure without friction of a prestressed elastic ring punch with a flat foundation on a prestressed half-space in the case of unequal roots of the characteristic (constitutive) equation [9]. The study is performed in general form for compressible and incompressible bodies for the theory of large (finite) initial strains and two variants of the theory of small initial strains with an arbitrary elastic potential, i.e., within the second approach. We believe that the initial stress–strain states in the punch and half-space are homogeneous and equal. The superscripts (1) and (2) are used to refer to the elastic punch and the half-space, respectively. A similar contact problem in the classical case, i.e., without initial stresses, is considered in [5].

1. Problem Statement and Basic Relations. Let a prestressed finite elastic ring punch of height H, inner R_1 and outer R_2 radii, and the geometric axis of symmetry coinciding with the y_3 axis of the cylindrical coordinate system (r, θ, y_3) and directed inside the half-space (Fig. 1) press on the half-space with a force P after occurrence of the initial strain state. We assume that the external load is applied only to the free end of the elastic punch, all points of the end of the punch displace along the axis of symmetry y_3 by the same value ε. We also assume that the surfaces outside the contact boundary remain free from external

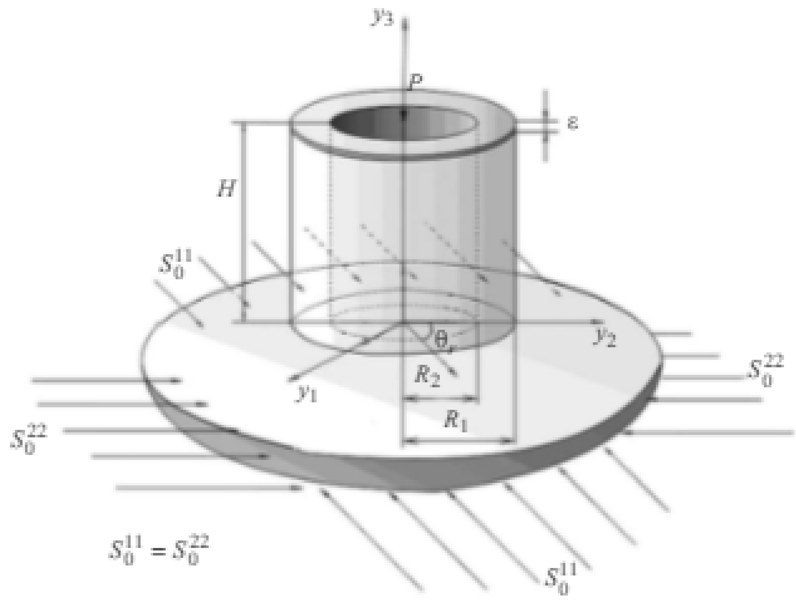

Fig. 1

forces, and, at the contact boundary, the displacement and stress are continuous. Also, $\lambda_i \, (i = 1, 2, 3)$ are the elongation coefficients that determine the displacement of the initial state, and S_0^{11}, S_0^{22} are the components of the symmetric tensor of the initial stresses.

Suppose that the initial states of the half-space and the punch are homogeneous, and the following relations are satisfied [10, 17]:

$$y_m = x_m + U_m^0, \quad U_m^0 = \delta_{mi}(\lambda_m - 1)\lambda_i^{-1} y_i \quad (i, m = \overline{1,3}).$$

Then the basic equation for displacements [10, 17] in compressible bodies has the form

$$L'_{m\alpha} U_\alpha = 0,$$

$$L'_{m\alpha} = \omega'_{ij\alpha\beta} \partial^2 / \partial y_\partial y_\beta \quad (i, m, \alpha, \beta = \overline{1,3}), \tag{1.1}$$

and in incompressible bodies, the incompressibility condition is satisfied:

$$L'_{m\alpha} U_\alpha + q'_{\alpha m} \partial p' / \partial y_\alpha = 0, \quad L'_{m\alpha} = \kappa'_{im\alpha\beta} \partial^2 / \partial y_\partial y_\beta, \tag{1.2}$$

$$q'_{ij} \partial U_j / \partial y_i = 0, \quad q'_{ij} = \lambda_i q_{ij} \quad (i, j, m, \alpha, \beta \ (\overline{1,3}).$$

The expressions for determining the components of the stress tensor for compressible and incompressible bodies are

$$Q'_{ij} = \omega'_{ij\alpha\beta} \frac{\partial U_\alpha}{\partial y_\beta}, \quad Q'_{ij} = \kappa_{ij\alpha\beta} \frac{\partial U_\alpha}{\partial y_\beta} + q'_{ij} p,$$

$$\omega'_{ij\alpha\beta} = \frac{\lambda_i \lambda_\beta}{\lambda_1 \lambda_2 \lambda_3} \omega_{ij\alpha\beta}, \quad \kappa'_{ij\alpha\beta} = \frac{\lambda_i \lambda_\beta}{\lambda_1 \lambda_2 \lambda_3} \kappa_{ij\alpha\beta}.$$

For homogeneous initial stresses, we assume that $S_0^{11} = S_0^{22} \neq 0, S_0^{33} = 0, \lambda_1 = \lambda_2 \neq \lambda_3$. Given these conditions, the solution of Eqs. (1.1), (1.2) is represented by a function χ that satisfies the equation

$$(\Delta_1 + \xi_2'^2 \partial^2 / \partial y_3^2)(\Delta_1 + \xi_3'^2 \partial^2 / \partial y_3^2)\chi = 0, \tag{1.3}$$

where $\Delta_1 = \partial^2 / \partial r^2 + r^{-1}\partial / \partial r$.

As noted above, we consider only the case of unequal roots ($\xi_2'^2 \neq \xi_3'^2$) of the characteristic equation corresponding to Eq. (1.3).

In the system of circular cylindrical coordinates (r, θ, z_i), where $z_i = v_i^{-1} y_3$, $v_i = \sqrt{n_i}$ $(i = 1, 2)$, $n_1 = \xi_2'^2$, $n_2 = \xi_3'^2$ such a statement corresponds to the boundary conditions

$$U_3^{(1)} = -\varepsilon, \quad Q_{3r}^{(1)} = 0 \quad (R_1 < r < R_2), \quad z_i = Hv_i^{-1} \ (i = 1, 2), \tag{1.4}$$

$$U_3^{(1)} = U_3^{(2)}, \quad \tilde{Q}_{33}^{(1)} = \tilde{Q}_{33}^{(2)}, \quad \tilde{Q}_{3r}^{(1)} = \tilde{Q}_{3r}^{(2)} = 0 \quad (R_1 < r < R_2), \quad z_i = 0 \quad (i = 1, 2), \tag{1.5}$$

$$\tilde{Q}_{33}^{(2)} = 0, \quad \tilde{Q}_{3r}^{(2)} = 0 \quad (0 < r < R_1, R_2 < r < \infty), \quad z_i = 0 \quad (i = 1, 2), \tag{1.6}$$

$$\tilde{Q}_{rr}^{(1)} = 0, \quad \tilde{Q}_{3r}^{(1)} = 0 \quad (0 \le z_i \le Hv_i^{-1}), \quad r = R_1, \quad r = R_2. \tag{1.7}$$

The equilibrium condition relating the subsidence of the end face and the equivalent load P has the form

$$P = -2\pi \int_{R_1}^{R_2} r Q_{33}^{(2)}(0, r) dr. \tag{1.8}$$

2. Solution Method. To determine the stress–strain state in a prestressed elastic ring punch, we use linearized equations [10, p. 78]. The expressions for the components of the displacement vector and the stress tensor for compressible and incompressible bodies follow from these equations. Then the general solution $\chi = \chi_1 + \chi_2$ for the case of unequal roots of the constitutive equation [10, formula (2.19)] is taken in the form

$$\chi_1 = A_0(r^2 - 2z_1^2) + C_0 z_1(3r^2 - 2z_1^2)$$

$$+ \{[A_k^{(1)} I_0(\gamma_k v_1 r) + A_k^{(2)} K_0(\gamma_k v_1 r)]C_k \sin(\gamma_k v_1 z_1) + [T_k^{(1)} J_0(\alpha_k r) + T_k^{(2)} Y_0(\alpha_k r)]\tilde{S}_2(\alpha_k z_1)\}M_k,$$

$$\chi_2 = A_0(r^2 - 2z_2^2) + C_0 z_2(3r^2 - 2z_2^2)$$

$$+ \{[B_k^{(1)} I_0(\gamma_k v_2 r) + B_k^{(2)} K_0(\gamma_k v_2 r)]C_k \sin(\gamma_k v_2 z_2) + [T_k^{(1)} J_0(\alpha_k r) + T_k^{(2)} Y_0(\alpha_k r)]\tilde{S}_3(\alpha_k z_2)\}M_k,$$

$$\Big(\tilde{S}_2(x) = \tilde{E}_k \sinh(x) + \tilde{F}_k \cosh(x); \quad \tilde{S}_3(x) = \tilde{N}_k \sinh(x) + \cosh(x);$$

$$T_k^{(1)} = -Y_1(\alpha_k R_1 R_2^{-1})(J_1(\alpha_k R_1 R_2^{-1}))^{-1} T_k^{(2)}, \quad A_k^{(1)} = K_1(\gamma_k v_1 R_1)(I_1(\gamma_k v_1 R_1))^{-1} A_k^{(2)},$$

$$B_k^{(1)} = K_1(\gamma_k v_2 R_1)(I_1(\gamma_k v_2 R_1))^{-1} B_k^{(2)}, \quad A_0 = (3C_0 H(v_1)^{-1} - \varepsilon n_1 n_2 (4(m_1 n_2 + m_2 n_1))^{-1}),$$

$$\tilde{E}_k = (1 + m_2)n_1((1 + m_1)n_2)^{-1} \coth(\alpha_k H v_1^{-1}), \quad \tilde{N}_k = -\coth(\alpha_k H v_2^{-1}),$$

$$\tilde{F}_k = -(1 + m_2)n_1((1 + m_1)n_2)^{-1}, \quad \tilde{S}_4(x) = \tilde{E}_k \cosh(x) + \tilde{F}_k \sinh(x)\Big),$$

where α_k, γ_k are the eigenvalues of problem (1.4)–(1.7), $M_k = \tilde{M}_k T_k^{(2)}$, $A_k^{(2)}, B_k^{(2)}, T_k^{(2)}, C_0, C_k$, $\tilde{M}_k =$ const, M_k are unknown quantities.

Then the stress–strain state in the prestressed ring punch for compressible (incompressible) bodies and uneven roots, taking into account (1.4)–(1.7), is represented as

$$U_r^{(1)} = -6C_0 r\theta_+ - \sum_{k=1}^{\infty}\Big\{\gamma_k^2\Big[v_1(K_1(v_1\gamma_k R_1)(I_1(v_1\gamma_k R_1))^{-1} I_1(v_1\gamma_k r) \quad K_1(v_1\gamma_k r))\widetilde{A}_k \cos(\gamma_k v_1 z_1)$$

$$+ v_2(K_1(v_2\gamma_k R_1)(I_1(v_2\gamma_k R_1))^{-1} I_1(v_2\gamma_k r) - K_1(v_2\gamma_k r))\widetilde{B}_k \cos(\gamma_k v_2 z_2)\Big] - \alpha_k^2 R_2^{-1}\Big(Y_1(\alpha_k r R_2^{-1})$$

$$-Y_1(\alpha_k R_1 R_2^{-1})J_1(\alpha_k r R_2^{-1})(J_1(\alpha_k R_1 R_2^{-1}))^{-1}\Big)\Big(v_1^{-1}\widetilde{S}_4(\alpha_k z_1) \quad v_2^{-1}\widetilde{S}_5(\alpha_k z_2)\Big)\Big\}M_k$$

$$U_3^{(1)} = 12C_0(m_1 z_1 n_1^{-1} + m_2 z_2 n_2^{-1}) - 4A_0\theta_8 + \sum_{k=1}^{\infty}\Big\{m_1\gamma_k^2(K_1(v_1\gamma_k R_1)(I_1(v_1\gamma_k R_1))^{-1} I_1(v_1\gamma_k r)$$

$$-K_1(v_1\gamma_k r))\widetilde{A}_k \cos(\gamma_k v_1 z_1) - \alpha_k^2 R_2^{-1}(Y_1(\alpha_k r R_2^{-1}) - Y_1(\alpha_k R_1 R_2^{-1})J_0(\alpha_k r R_2^{-1})(J_0(\alpha_k R_1 R_2^{-1}))^{-1})$$

$$\times\Big[m_1 n_1^{-1}\widetilde{S}_2(\alpha_k z_1) + m_2 n_2^{-1}\widetilde{S}_3(\alpha_k z_2)\Big]\Big\}M_k,$$

$$Q_{33}^{(1)} = C_{44}\Big\langle 12(1+m_1)l_1\Big[v_1^{-1} + s v_2^{-1}\Big]C_0 + \sum_{k=1}^{\infty}\Big\{\gamma_k^3\Big((1+m_1)l_1 v_1^2\widetilde{A}_k \sin(\gamma_k v_1 z_1)\Big[K_0(\gamma_k v_1 r) + K_1(\gamma_k v_1 R_1)$$

$$\times(I_1(\gamma_k v_1 R_1))^{-1} I_0(\gamma_k v_1 r)\Big] + (1+m_2)l_2 v_2^2\widetilde{B}_k \cos(\gamma_k v_1 z_1)$$

$$\times\Big[K_0(\gamma_k v_2 r) + K_1(\gamma_k v_2 R_1)(I_1(\gamma_k v_2 R_1))^{-1} I_0(\gamma_k v_2 r)\Big]\Big)$$

$$-\alpha_k^3 R_2^{-1}(Y_1(\alpha_k r R_2^{-1}) - Y_1(\alpha_k R_1 R_2^{-1})J_0(\alpha_k r R_2^{-1})(J_0(\alpha_k R_1 R_2^{-1}))^{-1})$$

$$\times\Big((1+m_1)l_1 v_1^{-1}\widetilde{S}_4(\alpha_k z_1) + (1+m_2)l_2 v_2^{-1}\widetilde{S}_5(\alpha_k z_1)\Big)\Big\}M_k\Big\rangle,$$

$$Q_{3r}^{(1)} = C_{44}\sum_{k=1}^{\infty}\Big\{\gamma_k^3\Big[(1+m_1)\widetilde{A}_k v_1 \sin(\gamma_k v_1 z_1)\Big(K_1(\gamma_k v_1 R_1)(I_1(\gamma_k v_1 R_1))^{-1} I_1(\gamma_k v_1 r) - K_1(\gamma_k v_1 r)\Big)$$

$$+(1+m_2)v_2\widetilde{B}_k \sin(\gamma_k v_2 z_2)(K_1(\gamma_k v_2 R_1)(I_1(\gamma_k v_2 R_1))^{-1} I_1(\gamma_k v_2 r) - K_1(\gamma_k v_2 r))\Big]$$

$$+\alpha_k^3 R_2^{-1}(Y_1(\alpha_k r R_2^{-1}) - Y_1(\alpha_k R_1 R_2^{-1})J_1(\alpha_k r R_2^{-1})(J_1(\alpha_k R_1 R_2^{-1}))^{-1})$$

$$\times[(1+m_1)n_1^{-1}\widetilde{S}_2(\alpha_k z_1) + (1+m_2)n_2^{-1}\widetilde{S}_3(\alpha_k z_2)]\Big\}M_k$$

$$\Big(\widetilde{S}_5(\alpha_k z_2) = \widetilde{N}_{2k}\cosh(\alpha_k z_2) + \sinh(\alpha_k z_2); \quad \widetilde{A}_k = A_k^{(2)}C_k; \quad \widetilde{B}_k = B_k^{(2)}C_k,$$

$$\theta_8 = m_1 n_1^{-1} + m_2 n_2^{-1}, \qquad \theta_+ = v_2^{-1} + 2v_1^{-1}\Big), \tag{2.1}$$

where $J_v(x), I_v(x)$ are Bessel functions of real and imaginary argument; $K_v(x)$ is the Macdonald function; $Y_v(x)$ is a Neumann function; the values of $C_{44}, l_1, l_2, m_1, m_2, s$ are determined from [9].

The stress–strain state in a prestressed half-space for unequal roots, taking into account (1.4)–(1.7) and $z_1 = 0$, is represented as follows [10, 23, 24]:

$$Q_{33}^{(2)} \quad \frac{\omega_3}{R_2 - R_1}\int_0^{\infty} F \;\eta\; J_0(\eta r)\, d\eta(\;,\;)$$

$$U_3^{(2)} \quad \frac{1}{\omega_2}\int_0^{\infty}\frac{F(\eta)}{\eta}J_0 \;\eta r\; d\eta(, \quad)$$

$$U_r^{(2)} = \omega_1\int_0^{\infty}\frac{F(\eta)}{\eta}J_1(\eta r)d\eta\,, \tag{2.2}$$

where

$$\omega_3 = c_{44}l_1(1+m_1)(s-s_0), \quad \omega_2 = v_1(m_1(s_3-s_2))^{-1}, \quad \omega_1 \quad =_0 s \quad -1$$

$$s = s_0 l_2 l_1^{-1}, \quad s_2 = m_2 v_1(m_1 v_2)^{-1}, \quad s_3 = (1+m_2)v_1((1+m_1)v_2)^{-1},$$

$F(\eta)$ are unknown functions.

Using solution (2.1) and satisfying the second condition in (1.4), the second condition in (1.7), we find the eigenvalues of problem (1.4)–(1.7) for $n_1 \neq n_2$:

$$\gamma_k = \frac{\pi k}{H}, \quad \alpha_k = \frac{\mu_k R_2}{R_1} \quad (J_1(\;_k\mu)Y_1(\;_k\mu R_2R_1^{-1}) - Y_1(\;_k\mu)J_1(\;_k\mu R_2R_1^{-1}) = \;).\; 0$$

From the boundary conditions (1.7), we get $C \;\; _0 C \;\; _k 0$. Also, satisfying the first condition in (1.5), we determine the unknown function $F(\eta)$ for (2.2) from the triple integral equations:

$$\int_0^{\infty}F(\eta)J_0(\eta r)d\eta = 0 \quad (R_2 < r < \infty),$$

$$\int_0^{\infty}\frac{F(\eta)}{\eta}J_0(\eta r)d\eta = f(r) \quad (R_1 < r < R_2),$$

$$\int_0^{\infty}F(\eta)J_0(\eta r)d\eta = 0 \quad (0 < r < R_1), \tag{2.3}$$

where $f(r) \quad \frac{\omega_2}{R_2}\left(\varepsilon \quad t_1\sum_{k=1}^{\infty}\alpha_k^2\left(\frac{Y_1(\alpha_k R_1R_2^{-1})}{J_1(\alpha_k R_1R_2^{-1})}J_0 \;\; \alpha_k R_2^{-1}r) + Y_0(\alpha_k R_2^{-1}r)\right)M_k\right),\; t_1 = \frac{m_1 - m_2}{n_2(1+m_1)}.$

We reduce the integral equations (2.3) to one, as in [3], using the discontinuous integral [21]:

$$\int_0^{\infty}\eta J_n(0.5\eta(R_2-R_1))J_n(0.5\eta(R_2+R_1))J_0(\eta r)d\eta$$

$$= \begin{cases} 0, & r^2 < R_1^2,\; r^2 > R_2^2, \\ 4P_{n-0.5}^{0.5}(\alpha)(\sqrt{2\pi}(R_1^2 - R_2^2)\sqrt[4]{1-\alpha^2})^{-1}, & R_1^2 < r^2 > R_2^2, \end{cases} \tag{2.4}$$

where $\alpha = 2(R_2^2 - 2r^2 + R_1^2)(R_2^2 - R_1^2)^{-1}$, $P_{n-0.5}^{0.5}(\alpha)$ is the attached Legendre function of the first kind [4].

We look for the function $F(\eta)$ in the following form [3]:

$$F(\eta) = R_2\sum_{n=0}^{\infty}W_{2n}J_{2n}(0.5\eta(R_2-R_1))J_{2n}(0.5\eta(R_2+R_1)), \tag{2.5}$$

where W_{2n} are unknown constants.

Substituting (2.5) into (2.3) and taking into account (2.4), we obtain the integral equation

$$\sum_{n=0}^{\infty} W_{2n} \int_0^{\infty} J_{2n}(0.5\eta(R_2-R_1)) J_{2n}(0.5\eta(R_2+R_1)) J_0(\eta r)\, d\eta = f(r). \tag{2.6}$$

For solving (2.6), we use the following expansion [2]:

$$\left(\frac{z}{2}\right)^{\gamma-\mu-\nu} J_\mu(az) J_\nu(bz) = \frac{a^\mu b^\nu}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} (\gamma+2m) J_{\gamma+2m}(z)$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\gamma+m+n)\, {}_2F_1(-n,-n+\mu,\nu+1; b^2 a^{-2})}{n!(m-n)!\Gamma(m+\mu+1)},$$

where ${}_2F_1(-n, -n+\mu, \nu+1, b^2 a^{-2})$ is a hypergeometric function; $\Gamma(z)$ is the gamma function [4].

Given the value of the integral [4]:

$$\int_0^{\infty} J_\mu(az) J_\nu(bz)\, dz = \frac{a^{-\nu-1} b^\nu \Gamma(0.5(\mu+\nu+1))}{\Gamma(\nu+1)\Gamma(0.5(\mu-\nu+1))}\, {}_2F_1(0.5(\nu+\mu+1),\ 0.5(\nu-\mu+1),\ \nu+1,\ b^2 a^{-2}). \tag{2.7}$$

After using (2.7) in (2.6), we multiply both sides of (2.6) by

$$\frac{T_{2n}(0.5\alpha)}{\sqrt{R_2^2-r^2}\sqrt{r^2-R_1^2}}\, r dr, \quad n = 0, 1, 2, \ldots,$$

where $T_{2n}(z)$ is the Chebyshev polynomial of the first kind [4].

Integrate it by r, taking into account the value of the integral

$$\int_{R_1}^{R_2} \frac{T_{2n}(0.5\alpha) T_{2k}(0.5\alpha)}{\sqrt{R_2^2-r^2}\sqrt{r^2-R_1^2}}\, r dr = \begin{cases} \pi/2, & n = k = 0, \\ \pi/4, & n = k > 0, \\ 0, & n \neq k. \end{cases}$$

Satisfying the second boundary condition (1.5), we have

$$\int_0^{\infty} F(\eta) \int_{R_1}^{R_2} r J_0(\mu_k r) J_0(\eta r)\, dr d\eta = \frac{C_{44}(R_2-R_1)(1+m_2)}{\omega_3 v_2} \alpha_k^2 t_1 \left[\frac{Y_1(\alpha_k R_1 R_2^{-1})}{J_1(\alpha_k R_1 R_2^{-1})} \tilde{O}_k^{(1)} - \tilde{O}_k^{(2)}\right] M_k$$

$$\left(\tilde{O}_1 = \frac{R_2}{\alpha_k}\left[R_1 J_1(\alpha_k R_1 R_2^{-1}) - R_2 J_1(\alpha_k)\right];\ \tilde{O}_2 = \frac{R_2}{\alpha_k}\left[R_1 Y_1(\alpha_k R_1 R_2^{-1}) - R_2 Y_1(\alpha_k)\right]\right). \tag{2.8}$$

Substituting (2.5) into (2.8), we obtain

$$\sum_{n=0}^{\infty} W_{2n} \int_0^{\infty} J_{2n}(0.5\eta(R_2-R_1)) J_{2n}(0.5\eta(R_2+R_1)) \int_{R_1}^{R_2} r J_0(\mu_k r) J_0(\eta r)\, dr d\eta$$

$$= \frac{C_{44}(R_2-R_1)(1+m_2)}{\omega_3 v_2} \alpha_k^2 t_1 \left[\frac{Y_1(\alpha_k R_1 R_2^{-1})}{J_1(\alpha_k R_1 R_2^{-1})} \tilde{O}_k^{(1)} - \tilde{O}_k^{(2)}\right] M_k. \tag{2.9}$$

To determine the constants M_i, W_{2i} ($i = 0, 1, 2, \ldots$) that appear in (2.1)–(2.3), we obtain an infinite system of linear algebraic equations consisting of (2.9) and (2.6). We solve this system by the method of reduction, taking into account that

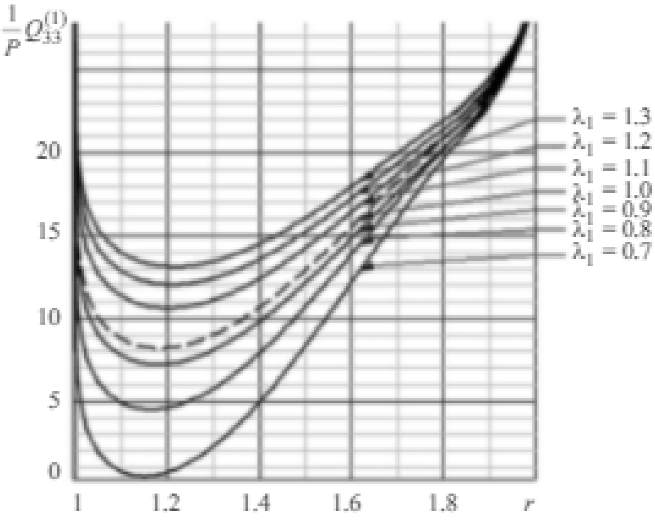

Fig. 2

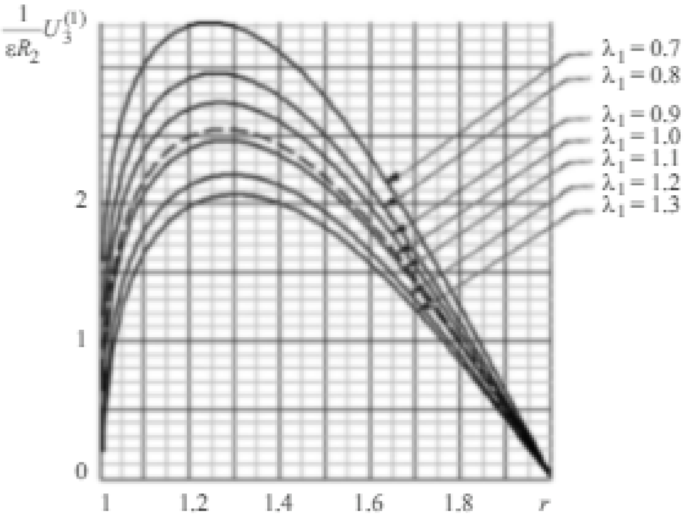

Fig. 3

$$W_0 = \omega_2 \varepsilon \pi (R_2 - R_1)(8\omega_3 R_2)^{-1} .$$

Using the equilibrium condition (1.8), we establish the relation between subsidence and the equivalent load P as

$$P = 2\omega_2 \omega_3 \varepsilon (\pi (R_2 - R_1))^{-1}.$$

Determining the unknown constants M_i, W_{2i} $(i = 0, 1, 2, \ldots)$ from the system of linear algebraic equations, we calculate the displacements and stresses in both the elastic punch and the elastic half-space by formulas (2.1)–(2.2). As a result, we represent the solution in the form of series in an infinite system of constants that are determined from a system of linear algebraic equations. Also, note that the coefficients of the system depend on the quantities that determine the structure of the elastic potential and the height H of the elastic punch.

3. Numerical Results. The numerical solution of the system of linear algebraic equations is found by the reduction method for the Treloar potential and the following parameter values: $R_1 = 1\cdot 10^{-2}$ m, $R_2 = 2\cdot 10^{-2}$ m, $\varepsilon = 10^{-4}$, $E = 8\cdot 10^{-5}$ MПa, $\lambda_1 = 0.7$, 0.8, 0.9, 1, 1.1, 1.2, 1.3, where $R_1 \le r \le R_2$. The solution algorithm is implemented as a Maple 15 program.

Figures 2 and 3 show the distributions of normal contact stress $\frac{1}{P}Q_{33}^{(1)}$ and displacement $-\frac{1}{\varepsilon R_2}U_3^{(1)}$ under the ring punch at the contact boundary in dimensionless coordinates. The dashed curve corresponds to without initial stresses ($\lambda_1 = 1$), and the solid curve to the half-space with initial stresses.

In the absence of initial stresses ($\lambda_1 = 1$), the graph of the distribution of contact stresses corresponds to the previously known solutions of the contact problem of the pressure of a ring punch on a half-space [5].

Conclusions. Based on numerical analysis, it can be claimed that under a constant external load, the initial stresses significantly affect the main contact characteristics (especially for incompressible bodies). In addition, the effect of the initial stresses on the stress–strain state of the elastic half-space into which a prestressed elastic ring punch is pressed, is that:

(1) the initial stresses in the half-space lead to a decrease in the stresses under compression ($\lambda_1 < 1$) and to their increase under tension ($\lambda_1 > 1$);

(2) in the case of displacements (Fig. 3), vice versa. The initial stresses in the half-space lead to an increase in the displacements under compression ($\lambda_1 < 1$) and to their decrease under tension ($\lambda_1 > 1$).

Thus, the results obtained taking into account the prestressed state in the case of contact interaction of an elastic punch and an elastic half-space can be used to control the contact stresses and displacements in the strength design of structures.

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