

CONTACT PROBLEM FOR TWO IDENTICAL STRIPS REINFORCED BY PERIODICALLY ARRANGED FASTENERS WITH INITIAL STRESSES*

S. Yu. Babich¹, N. N. Dikhtyaruk², and S. V. Degtyar²

The plane contact problem on the transfer of loading from a periodically arranged elastic finite pads to two identical elastic strips with initial stresses is solved using the linearized theory of elasticity. The study is carried out in a general form for the theory of large initial deformations and different variants of the theory of small initial deformations for an arbitrary elastic potential. The effect of the initial (residual) stresses on the distribution of the contact stresses over the interface with the periodically arranged elastic finite pads is investigated.

Keywords: contact problems, linearized theory of elasticity, elastic protective strip, initial (residual) tension, initial deformation

Introduction. One of the important factors having a strong effect on the reliability and durability of engineering facilities, mechanical structures, and machine parts is initial (residual) stresses. In real structures and machine parts, initial or residual stresses exist nearly always. The causes may be different. Often, the initial stresses in parts and structures are created specifically during their manufacture or assembly to compensate for the stresses that occur in structural elements and increase their strength characteristics.

One of the most common ways of transferring external forces is contact interaction; therefore, the study of this issue has been a very urgent problem for many years. At present, in our country and abroad, the study of contact interactions related to the transfer of load from elastic pads of various shapes and lengths to massive bodies is becoming increasingly relevant. The studies in this field were further developed in the monographs [1, 6]. The contact interaction between a prestressed half-plane and pads is studied in [1, 7, 9, 22]. The effect of the prestresses in an elastic strip on its contact interaction with elastic pads is studied in [3, 15]. The present study is novel in that it deals with the contact interaction between periodically arranged pads of finite length and two pre-stressed strips. This, in turn, expands the class of contact problems that can be formulated and solved using the linearized theory of elasticity.

1. Problem Statement. Basic Equations. Using the linearized theory of elasticity [1, 16, 18], we will formulate and solve the problem of load transfer from periodically arranged elastic pads of finite length to two identical prestressed strips clamped at one edge.

¹S. P. Timoshenko Institute of Mechanics, National Academy of Sciences of Ukraine, 3 Nesterova Str., Kyiv, Ukraine 03057. ²Khmelnytskyi National University, 11 Institutskaya Str., Khmelnytsky, Ukraine 29016; e-mail: mega-dihtyaruk@ukr.net. Translated from *Prikladnaya Mekhanika*, Vol. 55, No. 6, pp. 56–63, November–December 2019. Original article submitted September 11, 2018.

* This study was sponsored by the budget program “Support for Priority Areas of Scientific Research” (KPKVK 6541230).

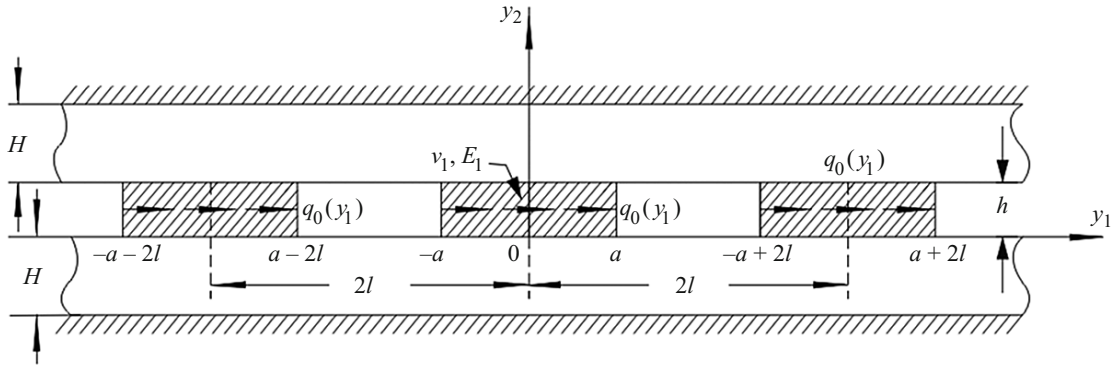


Fig. 1

Assume that two endless elastic strips are made of the same compressible or incompressible materials with an arbitrary elastic potential, and that these strips have the same initial (residual) stresses. It is assumed that they are joined at end segments $[-a + 2kl, a + 2kl]$ ($l > a, k = 0 \pm 1, \pm 2, \dots$) by periodically arranged elastic pads of small thickness h . The initial stress in the elastic pads are zero, the pads have elastic modulus E_1 and Poisson's ratio ν_1 and are in plane strain conditions. The thickness of the prestressed strips is H .

It is necessary to determine the effect of the initial stresses in the elastic strips on the distribution of the normal $p(y_1)$ and tangential $q(y_1)$ contact stresses at the interface between the elastic pads and the elastic strips, when they are subjected to a horizontal periodic load with a period $2l$ and intensity $q_0(y_1)$ (Fig. 1).

Since the problem is periodic, the influence of the initial (residual) stresses under each of the elastic thin pads will be the same; therefore, we can consider only one of them. Consider, for example, the pad that is located on the segment $[-a, a]$.

Note that the elastic pad bends like a regular beam in the vertical direction Oy_2 , and shrinks or stretches like a regular rod with finite stiffness that is in a uniaxial stress-strain state in the horizontal direction Oy_1 [4, 17]. Based on this, the following equations will be used to solve the problem:

$$\frac{du(y_1)}{dy_1} = \frac{1}{E_1 h} \int_{-a}^{y_1} [2q(\xi) - q_0(\xi)] d\xi \quad (-a < y_1 < a), \quad (1)$$

$$\frac{dv(y_1)}{dy_1} = 0, \quad \forall y_1 \in (-a < y_1 < a),$$

where $u(y_1), v(y_1)$ is the displacement of the boundary points of the stringer.

2. Transformation of Singular Integral Equations. The stringer-strip contact conditions have the form

$$u(y_1) = u_1(y_1), \quad v(y_1) = u_2(y_1), \quad (-a < y_1 < a), \quad (2)$$

where $u_1(y_1)$ and $u_2(y_1)$ are the displacements of boundary points of the elastic strip with initial (residual) stresses.

The boundary conditions at the ends of the elastic pad (stringer) due to the absence of external forces can be written as:

$$Q(y_1) = 0, \quad P(y_1) = 0, \quad M(y_1) = 0 \quad \text{at} \quad y_1 = \pm a,$$

where

$$Q(y_1) = \int_{-a}^{y_1} [q(\tau) - q_0(\tau)] d\tau, \quad |y_1| \leq a,$$

$$P(y_1) = \int_{-a}^{y_1} [p(\tau) - p_0(\tau)] d\tau, \quad |y_1| \leq a,$$

$$M(y_1) = \int_{-a}^{y_1} (y_1 - \tau)[p(\tau) - p_0(\tau)]d\tau, \quad |y| \leq a.$$

Here $Q(y_1)$ is the longitudinal force; $P(y_1)$ is the shear force; $M(y_1)$ is the bending moment in the cross-section of the stringer.

Taking into account (2) and the expressions for the displacements of the boundary points in the contact region $y_1 \in [-a, a]$ on the edge $y_2 = 0$ of the elastic strip with the initial (residual) stresses [15], we obtain a well-known system of integro-differential equations.

If we introduce a new function and make a replacement for this system,

$$X(\tau) = \tilde{p}(\tau) + i\tilde{q}(\tau), \quad (\tau = \xi, \eta), \quad \delta = \pi a / l, \quad (3)$$

then after some transformations we will obtain a singular integral equation with Hilbertian kernel [3, 4]:

$$\begin{aligned} & i\beta_1 X(\xi) + \int_{-\delta}^{\delta} X(\eta) \cot \frac{\xi - \eta}{2} d\eta - \int_{-\delta}^{\delta} \tilde{L}_{11}(\xi - \eta) X(\eta) d\eta \\ & - i \int_{-\delta}^{\delta} \tilde{L}_{12}(\xi - \eta) X(\eta) d\eta - \int_{-\delta}^{\delta} \tilde{L}_{22}(\xi - \eta) \bar{X}(\eta) d\eta + \beta_2 \int_{-\delta}^{\delta} [X(\eta) - \bar{X}(\eta)] d\eta \\ & = i[\beta_2 \tilde{Q}_1(\xi) + \lambda_4] d\tau - \frac{\beta_1}{2\pi} \quad (-\delta < \xi < \delta) \end{aligned} \quad (4)$$

with the boundary condition

$$\int_{-\delta}^{\delta} X(\eta) d\eta = \frac{i\pi}{l}, \quad (5)$$

where $\bar{X}(\eta)$ is a complex conjugate of $X(\eta)$,

$$\begin{aligned} \tilde{Q}_1(\xi) &= \int_{-\delta}^{\delta} \tilde{q}_0(\eta) d\eta, \quad Q_0 = \int_{-a}^a q(\eta) d\eta, \quad M_0 = \int_{-a}^a \xi p(\xi) d\xi, \\ \tilde{p}(\tau) &= p\left(\frac{l\tau}{\pi}\right) \frac{1}{Q_0}, \quad \tilde{q}(\tau) = q\left(\frac{l\tau}{\pi}\right) \frac{1}{Q_0}, \quad \tilde{q}_0(\tau) = q_0\left(\frac{l\tau}{\pi}\right) \frac{1}{Q_0}. \end{aligned} \quad (6)$$

The quantities $\tilde{L}_{ij}(\tau)$, β_i , λ_4 characterize the initial (residual) stress state and are determined for compressible and incompressible bodies in the case of specific elastic potentials separately for equal and nonequal roots of the constitutive equation [1]; $\tilde{Q}_1(\xi)$ is the distribution function of the tangential contact stresses along the contact line of the stringer and the elastic strip with the initial stresses.

Thus, the solution of the problem is reduced to the solution of the singular integral equation (4) with boundary condition (5).

This candidate solution of this equation is represented as a series of Jacobian functions [4, 6]:

$$X(\xi) = w(\xi) \sum_{n=1}^{\infty} \tilde{X}_n P_n^{(\alpha, \beta)} \left(\cot \frac{\delta}{2} \tan \frac{\xi}{2} \right) \quad (|\xi| < \delta), \quad (7)$$

where $\{P_n^{(\alpha, \beta)}(\cot \delta / 2 - \tan \xi / 2)\}_{n=0}^{\infty}$ is a Jacobi polynomial, orthogonal on the segment $[-\delta, \delta]$ with respect to weight

$$w(\xi) = \sec \frac{\xi}{2} \left(\sin \frac{\delta - \xi}{2} \right)^\alpha \left(\sin \frac{\delta + \xi}{2} \right)^\beta,$$

$$\alpha = -\frac{1}{2} - i\alpha_1, \quad \beta = -\frac{1}{2} + i\alpha_1, \quad \alpha_1 = \frac{\ln(3 - 4c_{44})}{2\pi}.$$

Here c_{44} is a parameter that determines the initial stress state in the strip.

Note that \tilde{X}_n is an infinite series of unknown complex coefficients that should be determined. To determine them, we substitute (7) into Eq. (4). Then, using the orthogonality properties of the Jacobi polynomials [5, 6] to determine the unknown quantities \tilde{X}_n , we obtain an infinite system of linear algebraic equations:

$$l_m \tilde{X}_m + \sum_{n=1}^{\infty} [D_{m,n}^{(1)} \tilde{X}_n + D_{m,n}^{(2)} \bar{\tilde{X}}_n] = -[D_m^{(0)} + D_m^{(1)} X_0 + D_m^{(2)} \bar{X}_0] \quad (m=1, 2, \dots), \quad (8)$$

where

$$l_m = \frac{4\pi \cot \frac{\delta}{2} \Gamma(m - \alpha) \Gamma(m - \beta)}{\cosh \pi \alpha_1 (m!)^2},$$

$$D_{m,n}^{(1)} = L_{m,n}^{(11)} + L_{m,n}^{(12)} + L_{m,n}^{(13)}, \quad D_{m,n}^{(2)} = L_{m,n}^{(22)} + L_{m,n}^{(23)}, \quad D_m^{(1)} = L_m^{(11)} + L_m^{(12)} - L_m^{(13)},$$

$$D_m^{(2)} = L_m^{(22)} + L_m^{(23)}, \quad D_m^{(0)} = \int_{-\delta}^{\delta} Q_1^*(\xi) w_1(\xi) P_{m-1}^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi,$$

$$L_{m,n}^{(j3)} = \lambda_1 \int_{-\delta}^{\delta} \left\{ \int_{-\delta}^{\xi} \left[(2-j)w(\eta) \cdot P_n^{(\alpha, \beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\eta}{2} \right) + (j-1)\bar{w}(\eta) \bar{P}_n^{(\alpha, \beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\eta}{2} \right) \right] d\eta \right\} w_1(\xi) \\ \times P_{m-1}^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi,$$

$$L_m^{(j3)} = \lambda_1 \int_{-\delta}^{\delta} \left\{ \int_{-\delta}^{\xi} [(2-j)w(\eta) + (j-1)\bar{w}(\eta)] d\eta \right\} w_1(\xi) P_{m-1}^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi,$$

$$L_{m,n}^{(1j)} = [(2-j) + i(j-1)] \int_{-\delta}^{\delta} \left\{ \tilde{L}_{ij}(\xi - \eta) w(\eta) \cdot P_n^{(\alpha, \beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi \right\} \\ \times w_1(\xi) \cdot P_{m-1}^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi, \quad j=1, 2,$$

$$L_{m,n}^{(22)} = \int_{-\delta}^{\delta} \left\{ \int_{-\delta}^{\delta} \tilde{L}_{22}(\xi - \eta) \bar{w}(\eta) d\eta \right\} w_1(\xi) P_n^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi,$$

$$L_m^{(22)} = \int_{-\delta}^{\delta} \left\{ \int_{-\delta}^{\delta} [\beta_2(\xi - \eta)^2 + \beta_3] \bar{w}(\eta) d\eta \right\} w_1(\xi) P_{m-1}^{(-\alpha, -\beta)} \left(\cot \frac{\delta}{2} \cdot \tan \frac{\xi}{2} \right) d\xi,$$

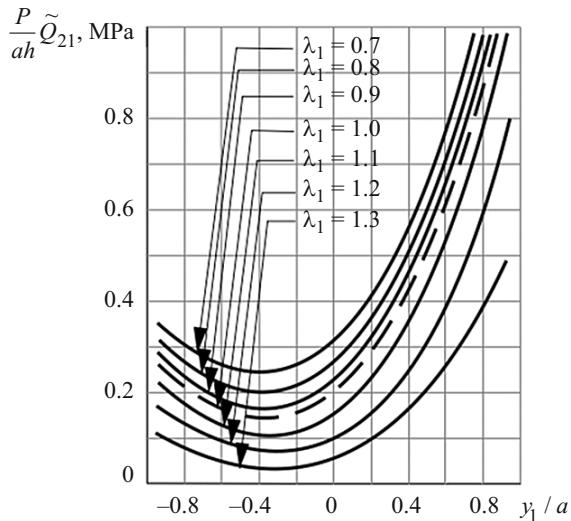


Fig. 2

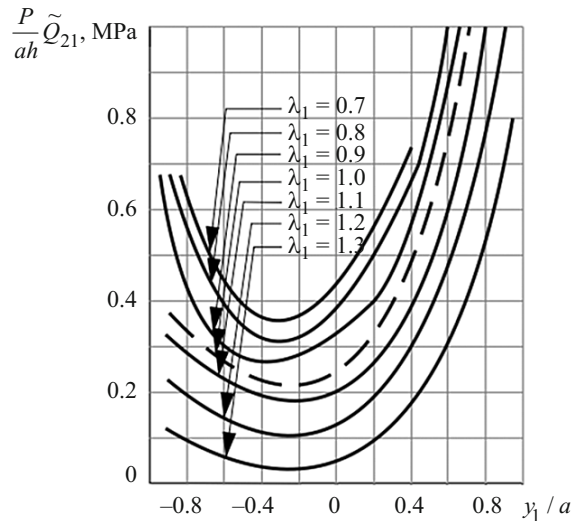


Fig. 3

where β_i, λ_1 are parameters that determine the initial stress state in the strip, $w_1(\xi) = \sec \frac{\xi}{2} \left(\sin \frac{\delta - \xi}{2} \right)^\alpha \left(\sin \frac{\delta + \xi}{2} \right)^\beta$.

To solve system (8) numerically, it is additionally necessary to determine the coefficient X_0 on the right-hand side of the system [4, 6, 18]. The coefficient X_0 can be found from boundary condition (5) using series (7). Then

$$X_0 = \frac{\cosh \pi \alpha_1 \cdot \cos \frac{\delta}{2}}{2l} i. \quad (9)$$

System (8) is quasiregular since the sums

$$L_m = l_m m^\varepsilon \sum_{n=1}^{\infty} \frac{1}{n^\varepsilon} [|D_{m,n}^{(1)}| + |D_{m,n}^{(2)}|] \quad (m=1,2,\dots) \quad (10)$$

and the free term $d_m = l_m^* m^\varepsilon \left[|D_m^0| + |D_m^{(1)}| + |D_m^{(2)}| \right]$, where $l_m^* = (ml_m)^{-1} = O(1)$, are of order of $O(m^{-(1/2)+\varepsilon})$ as $m \rightarrow \infty$. Such estimates make it possible to argue that the system is completely quasi-regular for arbitrary values of the physical and geometric parameters of a wide class of structural materials. This means that the coefficients of the matrix of the system rapidly decrease with increasing m and n from the diagonal elements and it can be solved with well-known numerical methods.

The present study makes it possible to draw a number of general conclusions regarding the influence of initial stresses on the distribution of contact forces under the pads interacting with the prestressed strips.

The above case confirms the assumption that the stresses under periodically arranged elastic pads reinforcing strips with and without initial stresses will be identical under the action of the same external forces. It can be easily verified by comparing the formulas for displacements in elastic strips [6] without initial stresses and strips [1, 4] with initial stresses under identical external forces.

The distance between the pads is known to be an important parameter of their interaction. However, it turned out that the distribution of contact stresses, in addition to the distance between the pads, is significantly affected by the initial stresses in the strips. These effects can be determined for a specific type of elastic potential using known parameters [1, 4]. According to [4–6], for the simplest potentials, the laws of variation in the displacements under the pads in the case of compressible (harmonic potential) and incompressible bodies (Bartenev–Khazanovich potential) [1] are shown in Figs. 2 and 3, respectively.

Here $\frac{P}{ah} \tilde{Q}_{21}$ are the dimensionless contact tangential stresses; λ_1 are the initial elongations along the axis Oy_1 , which determine the displacements in the initial state; P is the vertical load applied to the pad; $\tilde{Q}_{21} = q_0(y_1) \delta(y_1)$ is the tangential force acting on the pad, where $\delta(y_1)$ is the Dirac function.

The value $\lambda_1 = 1$ (dashed line) corresponds to the classical theory of elasticity and coincides with the results of [7]; $\lambda_1 = 0.7, 0.8, 0.9$ corresponds to the initial compressive stresses; $\lambda_1 = 1.1, 1.2, 1.3$ corresponds to the initial tensile stresses; y_1 / a is the dimensionless coordinate of the initial stress state in elastic strips with initial stresses.

It can be seen that the displacements under the pads increase significantly in the case of compression ($\lambda_1 < 1$) and decrease in the case of tension ($\lambda_1 > 1$). The qualitative picture is the same for both potentials. Note that a similar pattern of the influence of initial stresses on displacements was noted earlier for elastic punches [1].

It should be noted that the singularities of the stresses at the ends of the pads are similar to those in the classical case, i.e., in the absence of initial stresses [6]. Analysis of the behavior of the Cauchy type integral near the ends of the integration line, as well as the form of solution (7) (weight function) indicate an oscillating singularity of the contact stresses near the ends of the pad [16].

Conclusions. The plane problem of transferring the horizontal concentrated load from an elastic pad of finite length to two equal strips with initial stresses clamped along one edge has been solved using the linearized theory of elasticity. The solution has a general form for the theory of large initial deformations and different theories of small initial deformations for an arbitrary elastic potential. The problem has been reduced to a singular integral equation whose solution is represented as a series of Jacobi polynomials. After several transformations [6, 12] we have obtained a quasiregular infinite system of linear algebraic equations that can be solved by well-known numerical methods [6].

It has been established that the presence of initial stresses in elastic strips leads to a significant change in the distribution of the contact stresses; however, the contact stresses decrease significantly in the case of compression and increase in the case of tension, and the displacements increase significantly in the case of compression and decrease in the case of tension. The initial (residual) stresses have a stronger (quantitative) effect in highly elastic materials than in rigid materials, the qualitative effect being the same according to the results.

REFERENCES

1. O. M. Guz, S. Yu. Babich, and V. B. Rudnitskii, *Contact Interaction of Prestressed Bodies* [in Ukrainian], Vyshcha Shkola, Kyiv (1995).
2. A. N. Guz, S. Yu. Babich, and Yu. P. Glukhov, *Mixed Problems for an Elastic Base with Initial Stresses*, Lambert Academic Publishing, Saarbrücken (2015).
3. M. M. Dikhtyaruk, "Determining the influence function for an elastic strip with initial (residual) stresses," in: *Proc. 4th Int. Symp. on Tribo-Fatigue* (September 23–27, 2002, Ternopil) [in Ukrainian], Ternop. Derzh. Tekhn. Univ. im. Ivana Pulyuya, Ternopil (2002), pp. 426–431.
4. M. M. Dikhtyaruk, "Periodic contact problem for an elastic band with initial (residual) stresses," *Dop. NAN Ukrainy*, No. 3, 46–49 (2004).
5. V. B. Rudnitskii and N. N. Dikhtyaruk, "A prestressed elastic strip with elastic reinforcements," *Int. Appl. Mech.*, **38**, No. 11, 81–88 (2002).
6. V. S. Sarkisyan, *Contact Problems for Half-Planes and Strips with Elastic Pads* [in Russian], Izd. Yerevan. Univ., Yerevan (1983).
7. V. N. Akopyan, S. A. Mirzoyan, and S. M. Mkhitarian, "The problem of the contact between a broken stringer and an elastic infinite strip containing a vertical edge crack," *Int. Appl. Mech.*, **51**, No. 2, 176–186 (2015).
8. V. M. Aleksandrov, "Optimal control of linear systems with interval constraints," *Zh. Vychisl. Mat. Mat. Fiz.*, **55**, No. 5, 758–775 (2015).
9. S. Yu. Babich and Yu. P. Glukhov, "Bending a plate on prestressed elastic foundation under live static load," *Int. Appl. Mech.*, **53**, No. 3, 287–299 (2017).
10. S. Yu. Babich, A. N. Guz, and V. B. Rudnitsky, "Contact problems for prestressed elastic bodies and rigid and elastic punches," *Int. Appl. Mech.*, **40**, No. 7, 744–765 (2004).

11. S. V. Bosakov, "Finite integral transform method in static problems for inhomogeneous plates," *Int. Appl. Mech.*, **50**, No. 6, 651–663 (2014).
12. A. N. Guz, "Ultrasonic nondestructive method for stress analysis of structural members and near-surface layers of materials: focus on Ukrainian research (review)," *Int. Appl. Mech.*, **50**, No. 3, 231–252, (2014).
13. A. N. Guz, "Establishing the foundations of the mechanics of fracture of materials compressed along cracks (review)," *Int. Appl. Mech.*, **50**, No. 1, 1–57 (2014).
14. A. N. Guz, "Recognition of the achievements of the S. P. Timoshenko Institute of Mechanics by the world's scientific community," *Int. Appl. Mech.*, **51**, No. 1, 1–11 (2015).
15. N. N. Dikhtyaruk, "Equilibrium of a prestressed strip reinforced with elastic plates," *Int. Appl. Mech.*, **40**, No. 3, 290–296 (2004).
16. N. N. Dikhtyaruk, "Load transfer from the infinite stringer to the two jammed along one edge identical stripes with initial (residual) stresses," *Sci. J. of the Ternopil Nat. Techn. Univ.*, **83**, No. 3, 51–61 (2016).
17. V. B. Rudnitsky and N. N. Dikhtyaruk, "A prestressed elastic strip with elastic reinforcements," *Int. Appl. Mech.*, **38**, No. 11, 1354–1360 (2002).
18. V. B. Rudnitsky and N. N. Dikhtyaruk, "Interaction between an infinite stringer and two identical prestressed strips: contact problem," *Int. Appl. Mech.*, **53**, No. 2, 149–155 (2017).
19. N. A. Yaretskaya, "Three-dimensional contact problem for an elastic layer and a cylindrical punch with prestresses," *Int. Appl. Mech.*, **50**, No 4, 378–388 (2014).
20. N. A. Yaretskaya, "The impact of the initial (residual) stresses on the contact interaction of elastic cylindrical punch and an elastic layer," *Bulletin of NAS of Ukraine*, No. 1, 57–62 (2014).
21. N. F. Yaretskaya, "Contact problem for the rigid ring stamp and the half-space with initial (residual) stresses," *Int. Appl. Mech.*, **54**, No. 5, 539–543 (2018).
22. W. K. Yuan, J. M. Long, Y. Ding, and G. F. Wang, "Micro/nanocontact between a rigid ellipsoid and an elastic substrate with surface tension," *ASME. J. Appl. Mech.*, **84**, No 1, 011012-1–011012-7 (2017).