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(Online First)Contact Problem for a Pre-Stressed Annular Stamp and a Half-Space With Initial (Residual) Stresses (The Case of Equal Roots)

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ABSTRACT

The article is devoted to the study of the contact interaction of a prestressed ring stamp and a half-space (base) with initial (residual) stresses without taking friction forces into account. The problem is solved for the case of equal roots of the resolving equation. The study is presented in a general form for the theory of large initial (finite) deformations and two versions of the theory of small initial deformations in the framework of the linearized theory of elasticity for an arbitrary structure of the elastic potential.

There is assumed that the initial states of the elastic ring stamp and the elastic half-space remain homogeneous and equal. The study is carried out in the coordinates of the initial (residual) deformed state, which are interrelated with Lagrange coordinates (natural state). In addition, the influence of the ring stamp causes small perturbations of the basic elastic deformed state.

Also, it is assumed that the elastic ring stamp and the elastic half-space are made of different isotropic, transversal-isotropic or composite materials.

Numerical analysis is presented by the form of graphs of contact stresses and displacements of a potential of a harmonic type.

The influence of the initial (residual) stress on the contact interaction between the elastic ring stamp and the elastic half-space of the potentials of a particular structure is investigated.

Keywords: *linearized theory of elasticity; contact interaction, ring stamp; half-space; harmonic potential; elastic bodies; theory of large initial (finite) deformations; theory of small initial deformations.*

1. Introduction

Contact problems are an important part of the mechanics of a deformable solid and form the theoretical basis for calculations for the contact strength, stiffness and wear resistance of mobile and fixed joints.

The applied needs of natural science, modern technology and the latest technologies in recent decades associated with the necessity to predict the contact behavior of various designs, stimulated the development of various mathematical models and methods of contact mechanics of bodies with different properties [1-4].

One of the important factors in the contact interaction of bodies is the influence of initial (residual) stresses. Despite a significant achievement in the development of contact problems, nevertheless the issue of taking into account the initial (residual) stresses in the contact interaction has remained almost completely undeveloped until recently. There is known, that almost all elements of the construction have initial stress. It can be caused by various reasons, for example, by technological operations conducted in the manufacture of a variety of materials or by assembly of a structure. In the case of composite materials, the initial stresses, as a rule, correspond to stresses along the reinforcing elements. In the earth's crust, they are formed due to the action of gravitational forces and technical processes. They must be taken into account when solving the problems of deformation of soils (especially frozen ones). In addition, in elasto-plastic bodies, internal residual stresses can also be present after removal of loads.

In the general case, consideration of the initial (residual) stresses requires the using of the apparatus of the nonlinear theory of elasticity[4, 5], but for the sufficiently large initial (residual) loads, one can confine ourselves to its linearized version[6-10].

Linearized theory of elasticity for the bodies with initial (residual) stresses as the linearization of the nonlinear theory of elasticity[4, 5] was first proposed in monograph [11]. Also, in his work [12], the author, using considerations of a physical nature and not always strictly adhering to the principle of linearization of the nonlinear theory, developed the theory of incremental deformations for the bodies with initial stresses. Nevertheless, a simplified version of such theory based on a physical nature was considered by Cauchy (XIX century). Today the results of the works[6-7, 10, 13, 14] are completely based on the linearized theory of elastic bodies with initial (residual) stresses.

The fundamental results of the linearized theory of elasticity were obtained by academician Gusem A.N.[6–10, 13, 15, 16]. For the first time, he solved a number of contact problems for compressible and incompressible bodies by one of the most effective approaches for materials with an arbitrary form of elastic potential and homogeneous initial (residual) stresses. This approach is based on the theory of the function of a complex variable for plane problems and potential theory for spatial problems. Further development of the theory of contact interaction of bodies with initial (residual) stresses was obtained in the works[3, 6-10, 13–20]. A general analysis of the main methods and the best known results in all directions of the contact interaction of bodies with initial (residual) stresses is presented in review articles[14, 20].

Allowance of initial (residual) stresses within the linearized theory of elasticity leads to a new formulation of the problems of interaction of deformable solids, which significantly differ from the formulation of the problems of the classical theory of elasticity. Taken into account the problems when the initial (residual) stresses of the system of basic differential equations, the expressions of determining the components of the tensors of the stress-strain state and the structure of the boundary conditions differ from the corresponding systems of equations and expressions of the classical theory of elasticity, nevertheless, in their structure and nature they are similar to ordinary contact problems. Thus, from the above, it follows the possibility of using many fundamental results and methods of the linear theory of elasticity.

The first equations of the linearized theory of elasticity of deformable bodies[10] were obtained by linearizing the basic relations of the nonlinear theory, taking into account the physical characteristics of the materials; these results are obtained for small subcritical deformations in Lagrangian coordinates, which coincide with the Cartesian coordinates in the undeformed state. Later the main relations were written in curvilinear coordinates using the tensor analysis[21]; equations in displacements were also obtained, for which in a homogeneous subcritical state some methods for their solution are considered.

A modern analysis of the approaches to constructing theories and basic results that are applied to the three-dimensional linearized theory of elasticity of deformable bodies and the three-dimensional linearized theory of the propagation of elastic waves in bodies with initial (residual) stresses is presented by the generalizing publication, respectively [8]. With the using of approaches of the modern type [8], modern analysis of the results is performed for a wide range of problems of the linearized mechanics of deformed bodies, namely:

1) For problems of the contact interaction of elastic bodies with initial (residual) stresses [14, 20];
2) For the stability theory of the local equilibrium state of black rocks[15];
3) For exact solutions of plane mixed problems of linearized mechanics of deformable bodies [9];
4) For non-destructive ultrasonic methods for determining stresses in solids [16].

There are also a number of other generalizing publications about linearized mechanics. Moreover, the works mentioned above are only fully or partially related to the subject matter of this article. More widely the history of development and the range of problems of the linearized theory of elasticity are presented in [20].

Thus, the development of effective methods for calculating the stress-strain state with allowance for the initial (residual) deformations within the framework of the linearized theory of elasticity is an actual and important scientific and technical problem.

Today, in accordance to the problems related to contact problems for elastic bodies, results have been obtained on a wide range of issues. They are represented by works[6, 7, 13, 17, 18, 22, 23]. There are also a number of general publications [5, 24, 25], which are fully or partially related to the subject of this study. Despite significant achievements, the number of studies on the contact interaction of prestressed bodies is relatively small.

A rather detailed review of the work of rigid stamps (including ring ones) associated with contact pressure in the case of absence of initial stresses is given in the monograph[26].

The contact interaction of rigid and elastic stamps with prestressed bodies is presented in [3, 6, 7, 13, 17, 18, 22]. Moreover, either the elastic potentials of a particular structure are considered, but also the problem is considered in a general form for compressible (incompressible) bodies with the potential of an arbitrary structure on the basis of the linearized theory of elasticity.

The solution of an axisymmetric contact problem on the pressure of a rigid ring stamp of a complex configuration on an elastic layer with initial stresses is considered in[17]. The influence of initial stresses on the contact interaction of a rigid ring stamp on an elastic half-space with initial (residual) stresses is presented in[19].

In this paper, for the first time (residual) stresses, for the bodies with initial (residual) stresses[6, 7, 10, 13, 14], the problem of the pressure of a prestressed elastic ring stamp was first considered for a half-space with initial (residual) stresses without allowance for friction forces for equal roots resolving equation[7]. The investigation is carried out in a general form for compressible and incompressible bodies for the theory of large initial deformations and for two versions of the theory of small initial deformations for an arbitrary structure of the elastic potential.

2. Statement of the problem and basic relations.

2.1 Basic relationships

We will distinguish such states of bodies with initial (residual) stresses: natural (initial stresses absent), initial, and disturbed state. All quantities, the last, consist of the sum of the values of the initial state and the corresponding perturbations. Since the perturbations are assumed to be smaller than the corresponding values of the initial state, the investigations are carried out within the framework of the linearized theory of elasticity [6, 7, 10, 13, 14].

For the study, we use the coordinates of the initial deformed state (y_1, y_2, y_3), which are related to the Lagrange coordinates (x_1, x_2, x_3) (natural state): $y_i=\lambda_i x_i$ $(i=\overline{1,3})$. Here λ_i $(i=\overline{1,3})$ are the elongation coefficients that determine the movement of the initial state $\lambda_i=\text{const}$ $(i=\overline{1,3})$. The y_3 axis is directed along the normal to the contact area.

We assume that the initial states in the ring stamp and the half-space are homogeneous and equal, and the elastic potentials are twice continuously differentiable functions of the algebraic invariants of the Green's deformation tensor [7, 25]. In this case the relations are:

$$y_m = x_m + U_m^0, \quad U_m^0 = \delta_{mi}(\lambda_m - 1)\lambda_i^{-1} y_i .$$

Then the basic equation in displacements for compressible bodies has the form:

$$L'_{m\alpha}U_\alpha = 0, \quad L'_{m\alpha} = \omega'_{ij\alpha\beta}\,\partial^2/\partial y_i \partial y_\beta, \quad (i, m, \alpha, \beta = \overline{1,3}), \tag{1}$$

and for incompressible bodies together with the incompressibility condition:

$$L'_{m\alpha}U_\alpha + q'_{\alpha m}\,\partial p'/\partial y_\alpha = 0, \; L'_{m\alpha} = \kappa'_{im\alpha\beta}\,\partial^2/\partial y_i \partial y_\beta, \quad q'_{ij}\,\partial U_j/\partial y_i = 0, \; q'_{ij} = \lambda_i q_{ij}, \; (i, j, m, \alpha, \beta = \overline{1,3}). \tag{2}$$

Expressions for determining the components of the stress tensor for compressible and incompressible bodies are written as:

$$Q'_{ij} = \omega'_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta}, \; Q'_{ij} = \kappa_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta} + q'_{ij}\,p, \; \omega'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\omega_{ij\alpha\beta}, \kappa'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\kappa_{ij\alpha\beta},$$

where $\omega'_{im\alpha\beta}$, $\kappa'_{im\alpha\beta}$ - the components of the fourth order tensor of the elastic moduls,

$$p = (\lambda_1 q_1)^{-1}\left\{\left[\tilde{\kappa}_{1111} - \lambda_1 q_1 (\lambda_3 q_3)^{-1}\left(\tilde{\kappa}_{1133} + \tilde{\kappa}_{1313}\right)\right]\Delta_1 + \tilde{\kappa}_{3113}\,\partial^2/\partial y_3^2\right\}\partial^2/\partial y_3^2\,\tilde{\chi}$$

For homogeneous initial stresses $S_0^{11} = S_0^{22} \neq 0; \; S_0^{33} = 0; \; \lambda_1 = \lambda_2 \neq \lambda_3$, we express the roots of equations (1), (2) in terms of the roots of the differential equation:

$$(\Delta_1 + \xi_2'^2 \, \partial^2/\partial y_3^2)(\Delta_1 + \xi_3'^2 \, \partial^2/\partial y_3^2)\tilde{\chi} = 0, \tag{3}$$

where $\Delta_1 = \partial^2/\partial r^2 + r^{-1}\,\partial/\partial r$.

Taking into account the condition of the existence of a unique solution of the linearized theory of elasticity for compressible and incompressible bodies, two versions of the representation of the general solution (3) are possible:

1) The case of equal roots $(\xi_2'^2 = \xi_3'^2)$:

$$\tilde{\chi} = \tilde{\chi}_1 + y_3\tilde{\chi}_2, \left(\Delta_1 + \xi_2'^2 \, \partial^2/\partial y_3^2\right)\tilde{\chi}_1 = 0, \ \left(\Delta_1 + \xi_2'^2 \, \partial^2/\partial y_3^2\right)\tilde{\chi}_2 = 0 \tag{4}$$

2) The case of unequal roots $(\xi_2'^2 \neq \xi_3'^2)$:

$$\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2, \left(\Delta_1 + \xi_2'^2 \, \partial^2/\partial y_3^2\right)\tilde{\chi}_1 = 0, \ \left(\Delta_1 + \xi_3'^2 \, \partial^2/\partial y_3^2\right)\tilde{\chi}_2 = 0. \tag{5}$$

In this study we dwell in detail only he case of equal roots of the resolving equation (3).

2.2 Formulation of the problem.

Let the finite prestressed ring stamp with a flat base, whose geometric axis of symmetry coincides with the y3 axis of the cylindrical coordinate system (r, θ, y_3), which is directed into the half-space (Fig. 1) and presses on the half-space with the force P, after the initial deformed state. $R_1, R_2 -$ internal and external radii of the stamp. We assume that the external load is applied only to the free end of the resilient stamp. Under the action of the load, all points on the end of the stamp move in the direction of the symmetry axis y_3 by the same amount ε. We assume that the surfaces outside the contact region remain free from the influence of external forces, and in the contact zone of displacement and stress, they are continuous.

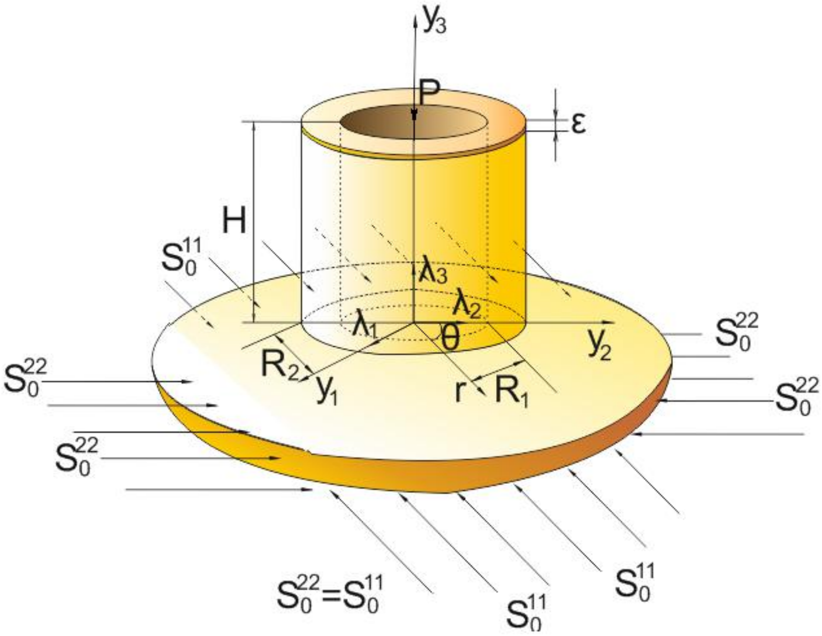

Fig. 1. Pressure of a prestressed ring stamp on a half-space with initial (residual) stresses.

The materials of the contacting bodies are assumed to be isotropic compressible or incompressible with an arbitrary structure of the elastic potential. In the case of orthotropic materials, it is assumed that the elastic-equivalent directions coincide with the directions of the coordinate axes (y_1, y_2, y_3).

The quantities that relate to the elastic stamp will be written with the superscript (1), and the quantities that relate to the pre-stressed half-space with the initial (residual) stresses - with the superscript (2).

2.3 Border conditions

In the system of circular cylindrical coordinates $(r,\ \theta,\ z_i)$, where $z_i = \nu_i^{-1} y_3, \quad \nu_i = \sqrt{n_i},\ (i = 1, 2),$

$n_1 = \xi_2'^2,\ \ n_2 = \xi_3'^2$ such a formulation corresponds to the boundary conditions:

1) at the end of the elastic stamp $z_1 = Hv_1^{-1}$

$$U_3^{(1)} = -\varepsilon, \quad Q_{3r}^{(1)} = 0 \quad (R_1 < r < R_2) \tag{6}$$

2) on the boundary of an elastic half-space in the region of contact $z_1=0$:

$$U_3^{(1)} = U_3^{(2)}; \quad \tilde{Q}_{33}^{(1)} = \tilde{Q}_{33}^{(2)}; \quad \tilde{Q}_{3r}^{(1)} = \tilde{Q}_{3r}^{(2)} = 0 \quad (R_1 < r < R_2) \tag{7}$$

3) on the boundary of the elastic half-space outside the contact region $z_1=0$:

$$\tilde{Q}_{33}^{(2)} = 0, \quad U_3^{(2)} = 0, \quad \tilde{Q}_{3r}^{(2)} = 0 \quad (0 < r < R_1 \quad R_2 < r < \infty) \tag{8}$$

4) on the outer side surface of the elastic stamp $r=R_2$:

$$\tilde{Q}_{rr}^{(1)} = 0, \quad \tilde{Q}_{3r}^{(1)} = 0 \quad (0 \le z_1 \le Hv_1^{-1}) \tag{9}$$

5) on the inner side surface of the elastic stamp $r=R_1$:

$$\tilde{Q}_{rr}^{(1)} = 0, \quad \tilde{Q}_{3r}^{(1)} = 0 \quad (0 \le z_1 \le Hv_1^{-1}) \tag{10}$$

The condition of equilibrium, which establishes the relationship between the draft of the butt and the resultant load P, is:

$$P = -2\pi \int_{R_1}^{R_2} r Q_{33}^{(2)}(0,r)\,dr \tag{11}$$

To determine the stress-strain state in an elastic ring stamp with initial (residual) stresses, we use the linearized equations (1) - (2), from which the expressions for the components of the displacement vector and the stress tensor for compressible and incompressible bodies follow. Then the general solution (4) for the case of equal roots $n_1 = n_2$ of the resolving equation (3) is taken in the form

$$\chi = \frac{\varepsilon v_1 z_1 (1 + v_1 z_1)}{(R_1 - R_2)(1 - m_2)} + \sum_{k=1}^{\infty} \left\langle \frac{v_1 z_1 \gamma_k \widetilde{B}_k (1 + v_1 z_1)}{R_1 - R_2} \left\{ \frac{1 + m_2}{1 - m_2} f_1(R_1, R_2) + \right.\right.$$
$$\left.\left. + \frac{I_0(\gamma_k v_1 R_1) - I_0(\gamma_k v_1 R_2)}{R_1 + R_2} \left(\frac{v_1 \theta_1}{1 - m_2} - \frac{3r^2 - 2z_1^2}{3} \right) \right\} + (\widetilde{A}_k + v_1 z_1 \widetilde{B}_k) I_0(\gamma_k v_1 r) \sin(\gamma_k v_1 z_1) + \right.$$
$$\left. + J_0(\alpha_k z_1)(\widetilde{S}_2(\alpha_k z_1) + v_1 z_1 \widetilde{S}_3(\alpha_k z_1)) \right\rangle N_k,$$

$$\chi_1 = \frac{\varepsilon v_1 z_1}{(R_1 - R_2)(1 - m_2)} + \sum_{k=1}^{\infty} \left\langle \frac{v_1 z_1 \gamma_k \widetilde{B}_k}{R_1 - R_2} \left\{ \frac{1 + m_2}{1 - m_2} f_1(R_1, R_2) + \right.\right.$$
$$\left.\left. + \frac{I_0(\gamma_k v_1 R_1) - I_0(\gamma_k v_1 R_2)}{R_1 + R_2} \left(\frac{v_1 \theta_1}{1 - m_2} - \frac{3r^2 - 2z_1^2}{3} \right) \right\} + \widetilde{A}_k I_0(\gamma_k v_1 r) \sin(\gamma_k v_1 z_1) + J_0(\alpha_k z_1) \widetilde{S}_2(\alpha_k z_1) \right\rangle N_k$$

$$\chi_2 = \frac{\varepsilon v_1 z_1}{(R_1 - R_2)(1 - m_2)} + \sum_{k=1}^{\infty} \left\langle \frac{v_1 z_1 \gamma_k \widetilde{B}_k}{R_1 - R_2} \left\{ \frac{1 + m_2}{1 - m_2} f_1(R_1, R_2) + \right.\right.$$
$$\left.\left. + \frac{I_0(\gamma_k v_1 R_1) - I_0(\gamma_k v_1 R_2)}{R_1 + R_2} \left(\frac{v_1 \theta_1}{1 - m_2} - \frac{3r^2 - 2z_1^2}{3} \right) \right\} + \widetilde{B}_k I_0(\gamma_k v_1 r) \sin(\gamma_k v_1 z_1) + J_0(\alpha_k z_1) \widetilde{S}_3(\alpha_k z_1) \right\rangle N_k,$$

where $\theta_1 = \dfrac{4Hm_1(1 + H)}{v_1^3} + \dfrac{R_1^2 + R_1 R_2 + R_2^2}{3} - \dfrac{2H^2}{n_1}$,

$$f_1(R_1,R_2)=-R_1I_0(\gamma_k v_1R_1)+\frac{\pi R_1}{2}L_0(\gamma_k v_1R_1)I_1(\gamma_k v_1R_1)-\frac{\pi R_1}{2}L_1(\gamma_k v_1R_1)I_0(\gamma_k v_1R_1)+$$
$$+R_2I_0(\gamma_k v_1R_2)-\frac{\pi R_2}{2}L_0(\gamma_k v_1R_2)I_1(\gamma_k v_1R_2)+\frac{\pi R_2}{2}L_1(\gamma_k v_1R_2)I_0(\gamma_k v_1R_2),$$

$$\widetilde{S}_2=\widetilde{E}_k sh(\alpha_k z_1)+\widetilde{F}_k ch(\alpha_k z_1),\quad \widetilde{S}_3=sh(\alpha_k z_1)+\widetilde{M}_k ch(\alpha_k z_1),\quad \widetilde{M}_k=-cth(\alpha_k Hv_1^{-1}),\quad \widetilde{F}_k=-s_0\alpha_k^{-1},$$

$$\widetilde{E}_k=Hcth^2(\alpha_k Hv_1^{-1})-\widetilde{F}_k cth(\alpha_k Hv_1^{-1})-H,\quad \widetilde{A}_k=\left(\frac{Rs_0H^2(I_0(\gamma_k v_1R_1)-I_0(\gamma_k v_1R_2))}{2n_1(R_2^2-R_1^2)I_1(\gamma_k v_1R)}-\frac{s_0}{v_1\gamma_k^2}+\frac{H}{2}\right)\widetilde{B}_k,$$

$$\widetilde{B}_k=\frac{\alpha_k}{2}(R_2^2-R_1^2)J_0(\alpha_kR)\{H\alpha_k(\widetilde{c}_0 sh(\alpha_k Hv_1^{-1})+\widetilde{c}_1(1-ch(\alpha_k Hv_1^{-1}))(1-s_0 sh(\alpha_k Hv_1^{-1}))+$$
$$+(1-ch^2(\alpha_k Hv_1^{-1}))(v_1(\widetilde{c}_0+\widetilde{c}_2-\widetilde{c}_1)-\widetilde{c}_0s_0)\}/\{\gamma_k(ch^2(\alpha_k Hv_1^{-1})-1)[H(1-\widetilde{c}_0+H(3\widetilde{c}_0+2\widetilde{c}_2-$$
$$-4\widetilde{c}_1))(I_0(\gamma_k v_1R_1)-I_0(\gamma_k v_1R_2))+\widetilde{c}_2n_1(R_2^2-R_1^2)I_0(\gamma_k v_1R)]\},$$

$$R=(R_2-R_1)\mathrm{H}(r-R_1)-(R_2-R_1)\mathrm{H}(r-R_2)+\delta(R_1)(2R_1-R_2)+\delta(R_1)R_2,$$

$\mathrm{H}(x)-$ the Heaviside function, $\delta(x)-$ the Dirac function, $L_v(x)-$ the modified Struve function, $J_v(x),\ I_v(x)-$ the Bessel functions of the real and imaginary argument, respectively.

$$m_1=\begin{cases}(\omega'_{1111}n_1-\omega'_{3113})(\omega'_{1133}+\omega'_{1313})^{-1};\\ \lambda_1q_1n_1(\lambda_3q_3)^{-1};\end{cases}\quad m_2=\begin{cases}(\omega'_{1133}-\omega'_{1313})(\omega'_{1133}+\omega'_{1313})^{-1},\\ 1,\end{cases}\quad s_0=\frac{(1+m_2)}{(1+m_1)},$$

$$\widetilde{c}_0=\begin{cases}\omega'_{1111}\omega'^{-1}_{1122},\\ \lambda_1q_1(\kappa'_{1133}+\kappa'_{1313})(\lambda_3q_3\kappa'_{1122})^{-1},\end{cases}\quad \widetilde{c}_1=\begin{cases}m_1\lambda_3\omega'_{1133}(\omega'_{1122}n_1)^{-1},\\ (\kappa'_{1133}m_1-\kappa'_{3113})(\kappa'_{1122}n_1)^{-1},\end{cases}\quad \widetilde{c}_2=\begin{cases}\lambda_3\omega'_{1133}(m_2-1)(\omega'_{1122}n_1)^{-1},\\ -\kappa'_{3113}(\kappa'_{1122}n_1)^{-1},\end{cases}$$

Then we obtain formulas for the displacement components for compressible and incompressible bodies:

$$U_r^{(1)}=-\sum_{k=1}^{\infty}\Big\{6\tilde{C}_0^{(k)}r\left(v_1^{-1}+2z_1\right)+\gamma_k v_1I_1(v_1\gamma_k r)\left[\left(\tilde{A}_k+v_1z_1\tilde{B}_k\right)\gamma_k\cos(\gamma_k v_1z_1)+\tilde{B}_k\sin(\gamma_k v_1z_1)\right]-$$
$$-\alpha_kJ_1(\alpha_k r)\left[\alpha_k v_1^{-1}\left(\tilde{S}_4(\alpha_k z_1)+v_1z_1\tilde{S}_5(\alpha_k z_1)\right)-\tilde{S}_3(\alpha_k z_1)\right]\Big\}N_k$$

$$U_3^{(1)}=\frac{\varepsilon}{R_1-R_2}+\sum_{k=1}^{\infty}\Big\{12m_1\tilde{C}_0^{(k)}z_1v_1^{-1}\left(v_1^{-1}+z_1\right)+(1-m_2)v_1^{-1}\left[\tilde{A}_0^{(k)}+3\tilde{C}_0^{(k)}\left(r^2-2z_1^2\right)\right]+$$
$$+\gamma_kI_0(\gamma_k v_1r)\left[\left(\tilde{A}_k+v_1z_1\tilde{B}_k\right)m_1\gamma_k\sin(\gamma_k v_1z_1)+(1-m_2)\tilde{B}_k\cos(\gamma_k v_1z_1)\right]-$$
$$-\alpha_kn_1^{-1}J_0(\alpha_k r)\left[m_1\alpha_k\left(\tilde{S}_2(\alpha_k z_1)+z_1v_1\tilde{S}_3(\alpha_k z_1)\right)+(m_2-1)v_1\tilde{S}_5(\alpha_k z_1)\right]\Big\}N_k \tag{12}$$

And the components of the stress vector for $y_3=const$ and $r=const$, respectively:

$$Q_{33}^{(1)}=C_{44}\sum_{k=1}^{\infty}\Big\{12\tilde{C}_0^{(k)}\left[(1+m_1)l_1\left(v_1^{-1}+z_1\right)+(1+m_2)l_2z_1\right]+\gamma_k^2v_1^2I_0(\gamma_k v_1r)\left[(1+m_1)l_1\gamma_k(\tilde{A}_k+\right.$$
$$\left.+v_1z_1\tilde{B}_k)\cos(\gamma_k v_1z_1)+(1+m_2)l_2\tilde{B}_k\sin(\gamma_k v_1z_1)\right]-$$
$$-\alpha_k^2J_0(\alpha_k r)\left[(1+m_1)l_1\alpha_k v_1^{-1}\left(\tilde{S}_4(\alpha_k z_1)+v_1z_1\tilde{S}_5(\alpha_k z_1)\right)+(1+m_2)l_2\tilde{S}_3(\alpha_k z_1)\right]\Big\}N_k$$

$$Q_{3r}^{(1)}=C_{44}\sum_{k=1}^{\infty}\Big\{-6\tilde{C}_0^{(k)}r(1+m_2)v_1^{-1}+\gamma_k^2v_1I_1(\gamma_k v_1r)\left[(1+m_1)\gamma_k\left(\tilde{A}_k+v_1z_1\tilde{B}_k\right)\sin(\gamma_k v_1z_1)-\right.$$
$$\left.-(1+m_2)\tilde{B}_k\cos(\gamma_k v_1z_1)\right]+$$
$$+\alpha_k^2v_1^{-1}J_1(\alpha_k r)\left[\alpha_k(1+m_1)\left(\tilde{S}_2(\alpha_k z_1)+v_1z_1\tilde{S}_3(\alpha_k z_1)\right)+(1+m_2)\tilde{S}_5(\alpha_k z_1)\right]\Big\}N_k \tag{13}$$

$$Q'_{rr}=-D_{44}\sum_{k=1}^{\infty}\Big\{6\tilde{C}_0^{(k)}\Big[v_1^{-1}\big(1+\tilde{c}_0-2\tilde{c}_1\big)+\big(3+\tilde{c}_0-4\tilde{c}_1+2\tilde{c}_2\big)z_1\Big]+$$
$$+\gamma_k v_1\Big(\gamma_k v_1 I_0(\gamma_k v_1 r)\Big[\gamma_k\big(\tilde{c}_0-\tilde{c}_1\big)\big(\tilde{A}_k+z_1v_1\tilde{B}_k\big)\cos(\gamma_k z_1 v_1)+$$
$$+\big(\tilde{c}_0-\tilde{c}_1+\tilde{c}_2\big)\tilde{B}_k\sin(\gamma_k z_1 v_1)\Big]+\big(1-\tilde{c}_0\big)I_1(\gamma_k v_1 r)r^{-1}\Big[\gamma_k\cos(\gamma_k z_1 v_1)\big(\tilde{A}_k+z_1v_1\tilde{B}_k\big)+\tilde{B}_k\sin(\gamma_k z_1 v_1)\Big]\Big)-$$
$$-\alpha_k\Big(\alpha_k J_0(\alpha_k r)\Big[\alpha_k v_1^{-1}\big\{\tilde{c}_0\tilde{S}_4(\alpha_k z_1)-$$
$$-\tilde{c}_1\tilde{S}_2(\alpha_k z_1)\big\}+\big(\tilde{c}_0-\tilde{c}_1+\tilde{c}_2\big)\tilde{S}_3(\alpha_k z_1)\Big]+\big(1-\tilde{c}_0\big)J_1(\alpha_k r)r^{-1}\Big[\alpha_k v_1^{-1}\tilde{S}_4(\alpha_k z_1)+\tilde{S}_3(\alpha_k z_1)\Big]\Big)\Big\}N_k,$$

where $\tilde{S}_4=\tilde{E}_k ch(\alpha_k z_1)+\tilde{F}_k ch(\alpha_k z_1)$, $\tilde{S}_5=ch(\alpha_k z_1)+\tilde{M}_k sh(\alpha_k z_1)$,

$$\tilde{A}_0^{(k)}=\frac{v_1}{(R_1-R_2)(1-m_2)}\sum_{k=1}^{\infty}\left\{(1+m_2)f_1(R_1,R_2)+\frac{v_1\theta_1(I_0(\gamma_k v_1 R_1)-I_0(\gamma_k v_1 R_2))}{R_1+R_2}\right\}\gamma_k\bar{B}_k,$$

$$\tilde{C}_0^{(k)}=\frac{v_1}{3(R_2^2-R_1^2)}\sum_{k=1}^{\infty}\{I_0(\gamma_k v_1 R_1)-I_0(\gamma_k v_1 R_2)\}\gamma_k\bar{B}_k.$$

$$D_{44}=\begin{cases}\omega'_{1122},\\ \kappa'_{1122}.\end{cases}\quad C_{44}=\begin{cases}\omega'_{1313},\\ \kappa'_{1313}.\end{cases}$$

$$l_1=\begin{cases}\omega'^{-1}_{1313}(\omega'_{1331}+(\omega'_{1313}-\omega'_{1331})(\omega'_{1133}+\omega'_{1313})(\omega'_{1111}n_1+\omega'_{1133})^{-1});\\ \kappa'^{-1}_{1313}(\kappa'_{1331}+\lambda_3q_3(\kappa'_{1313}-\kappa'_{1331})(\lambda_3q_3+\lambda_1q_1n_1)^{-1});\end{cases}$$

$$l_2=\begin{cases}(\omega'_{3333}(m_1+m_2-1)-\omega'_{1133}n_1)(n_1\omega'_{1313}(1+m_2))^{-1},\\ (\kappa'_{3333}m_1+(\lambda_1^{-1}q_1^{-1}\lambda_3q_3\kappa'_{1111}-2\kappa'_{1133}-\kappa'_{1313})n_1-3\lambda_1^{-1}q_1^{-1}\lambda_3q_3\kappa'_{3113})(2n_1\kappa'_{1313})^{-1}.\end{cases}$$

Stress-strain state in the elastic half-space with initial (residual) voltage equal to root ($n_1=n_2$) is defined by harmonic function Hankel integrals. Having satisfied the third condition (7), the third - (8), after a number of transformations we have

$$U_3^{(2)}=-\frac{1}{\omega_3}\int_0^{\infty}\frac{F(\eta)}{\eta}J_0(\eta r)d\eta,\quad U_r^{(2)}=\omega_1\int_0^{\infty}\frac{F(\eta)}{\eta}J_1(\eta r)d\eta,$$

$$Q_{33}^{(2)}=\frac{\omega_3}{R_2-R_1}\int_0^{\infty}F(\eta)J_0(\eta r)d\eta,\quad Q_{3r}^{(2)}=0, \tag{14}$$

where $\omega_3=C_{44}l_1(1+m_1)(s-s_0)$, $\omega_1=s_0-1$, $s=s_0\frac{l_2}{l_1}$.

3. Materials and Methods.

Using the solution for the cylinder (12) – (13) and satisfying the second condition (6), the second condition (9) and the second condition (10), we find the eigenvalues of the problem (6) – (11) (for the case of equal roots $n_1 = n_2$):

$$\gamma_k=\frac{\pi(2k+1)}{H},\quad \alpha_k=\frac{\mu_k}{R}\quad (J_1(\mu_k)=0). \tag{15}$$

Having satisfied the first conditions (7) and (8), we define the unknown function F(η) for (14) from the pairwise integral equations (for equal roots):

$$\int_0^{\infty}F(\eta)J_0(\eta r)d\eta=0\quad(0<r<R_1,R_2<r<\infty), \tag{16}$$

where $$f(r)=\frac{\omega_3\varepsilon}{R_2-R_1}+\sum_{k=1}^{\infty}\Big\langle(\tilde{A}_0^{(k)}+3r^2\tilde{C}_0^{(k)})\frac{1-m_2}{v_1}+\gamma_k I_0(\gamma_k v_1 r)(1-m_2)\tilde{B}_k-$$

$$-\frac{\alpha_k}{n_1} J_0(\alpha_k r)(m_1\alpha_k \tilde{F}_k + v_1(m_2-1))\Big\rangle N_k .$$

The application of the inversion formula to (16) leads to an integral equation of the Fredholm's type of the second kind with respect to the function F(η), that is

$$\frac{F(\eta)}{\eta} = \frac{2\omega_3}{\pi(R_2-R_1)}\Bigg\langle \varepsilon + \sum_{k=1}^{\infty}\Bigg[(1+m_2)f_1(R_1,R_2) + \frac{I_0(\gamma_k v_1 R_1) - I_0(\gamma_k v_1 R_2)}{R_2+R_1}(v_1\theta_1 - (1-m_2)(R_2^2-R_1^2))\gamma_k \tilde{B}_k \psi_0(\eta,0) +$$

$$+(R_1-R_2)\left\{\gamma_k \psi_0(\eta, i\gamma_k v_1(R_2-R_1))(1-m_2)\tilde{B}_k + \frac{\alpha_k}{n_1}\psi_0(\eta,\mu_k)((1-m_2)v_1 - s_0 m_1)\right\}\Bigg] N_k \Bigg\rangle, \quad (17)$$

where где

$$\psi_n(x,y) = \int_0^1 t^n \cos xt \cos yt dt, \quad \psi_0(x,y) = \frac{x\sin x\cos y - y\sin y\cos x}{x^2-y^2}, \quad \psi_0(x,0) = \frac{\sin x}{x},$$

$$\psi_0(x,iy) = \frac{x\sin x chy + y shy \cos x}{x^2+y^2}, \quad \psi_0(0,iy) = \frac{shy}{y} .$$

Having satisfied the second boundary condition (7), we have

$$-R_1 I_1(\gamma_k v_1 R_1))\tilde{A}_k - \alpha_k(R_2 J_1(\alpha_k R_2) - R_1 I_1(\alpha_k R_1))(\frac{l_1}{v_1}(1+m_1)\alpha_k \tilde{E}_k + (1+m_2) l_2 \tilde{M}_k)\Big\rangle N_k .$$

Having satisfied the first two boundary conditions (7), taking into account the orthogonality of the Bessel functions $J_0(\mu_k\rho)$ and the values of the integrals

$$\int_0^\infty \eta\psi_n(\eta,\mu_k)d\eta \int_0^1 \rho J_0(\eta\rho) J_0(\mu_k\rho)d\rho = \psi_n(\mu_n,\mu_k), \quad \int_0^\infty \psi_0(\eta,\mu_k) J_1(\eta)d\eta = \psi_0(0,\mu_k),$$

$$\int_0^\infty \psi_0(\eta,0)J_1(\eta)d\eta = 1, \quad \int_{R_1}^{R_2} \rho J_0(\mu_k\rho)J_0(\mu_n\rho)d\rho = \frac{1}{\mu_k^2-\mu_n^2}(R_1\mu_n J_0(\mu_k R_1)J_1(\mu_n R_1) - R_1\mu_k J_0(\mu_n R_1)J_1(\mu_k R_1) -$$

$$-R_2\mu_n J_0(\mu_k R_2)J_1(\mu_n R_2) + R_2\mu_k J_0(\mu_n R_2)J_1(\mu_k R_2)) .$$

to determine the constants N_i (i = 0,1,2, ...) that occur in (12) - (14) and (16), we obtain an infinite system of algebraic equations

$$\sum_{n=0}^{\infty} \tau_{kn}\chi_n = \beta_k \quad (k = 0,1,2,...) . \quad (18)$$

We represent the coefficients of the system in the form

$$\tau_{kn} = \left\{(1+m_2)f_1(R_1,R_2) + \frac{I_0(\gamma_n v_1 R_1) - I_0(\gamma_n v_1 R_2)}{R_1+R_2}(v_1\theta_1 - (R_2^2-R_1^2)(1-m_2))\right\}\gamma_n \widetilde{B}_k +$$

$$+(R_1-R_2)\left\{\gamma_n \widetilde{B}_k (1-m_2)\psi_0(0, i\gamma_n v_1 R) + \frac{\alpha_n}{n_1}\psi_0(0,\mu_n)((1-m_2)v_1 + m_1 s_0)\right\} -$$

$$-C_{44}\{2(1+m_1)l_1\gamma_n \widetilde{B}_k (I_0(\gamma_n v_1 R_1) - I_0(\gamma_n v_1 R_2)) + \gamma_n^2(1+m_1)l_1 v_1(R_2 I_1(\gamma_n v_1 R_2) -$$

$$-R_1 I_1(\gamma_n v_1 R_1))\overleftarrow{A}_k - \alpha_n(R_2 J_1(\alpha_n R_2) - R_1 J_1(\alpha_n R_1))((1+m_1)l_1 v_1^{-1}\alpha_n \bar{E}_k + (1+m_2)l_2 \widetilde{M}_k)\},$$

$$\beta_k = \frac{2\omega_3\varepsilon}{\pi R_1 R_2} .$$

Using the equilibrium condition (11), we establish the relationship between the draft and the resultant load P in the form

$$P = -\frac{4\omega_3^2\varepsilon}{(R_2-R_1)R_2R_1}\sum_{k=1}^{\infty}\{[(I_0(\gamma_k v_1 R_1) - I_0(\gamma_k v_1 R_2))(v_1\theta_1 - (R_2^2-R_1^2)(1-m_2))\gamma_k -$$

$$-(1-m_2)v_1^{-1}sh(\gamma_k v_1(R_2-R_1))\big]\widetilde{B}_k+v_1^{-1}\sin\mu_k((m_2-1)v_1-s_0m_1)\big\}N_k\;.$$

Having determined the unknown constants N_i (i=0,1,2,..) from the system (18), we calculate the displacements and stresses both in the elastic stamp and in the elastic half-space by the formulas (12) - (14). As a result, we represent the solution in the form of series through an infinite system of constants, which are determined from the system of linear algebraic equations (18). Moreover, in the system (18) the coefficients β_k and τ_{kn} depend on the quantities that determine the structure of the elastic potential, the height of the elastic stamp H, and the free terms depend only on the roots of the characteristic equation n_1, n_2.

4. Results.

4.1 Numerical analysis.

The numerical solution of the system (18) is carried out in the work by the reduction method for the harmonic potential at such values of the parameters: R_1=1.0, R_2=2.0, $\varepsilon = 10^{-5}$, E =3.92, σ =0.47, λ_1= 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3,, where $R_1 \le r \le R_2$. The algorithm for solving this problem is implemented in the form of a computer program in the package Maple 17.

On Fig. 2, 3, there are, respectively, the distribution of the normal contact voltage $\frac{1}{\varepsilon}Q_{33}$ and displacement $\frac{1}{\varepsilon}U_3$ under the ring stamp in the contact zone in dimensionless coordinates. Moreover, the values of the elongation coefficients λ_1 correspond to the curves from the bottom up ascending λ_1. The dotted curves correspond to a half-space without initial stresses (λ_1 = 1), and solid curves correspond to initial (residual) stresses.

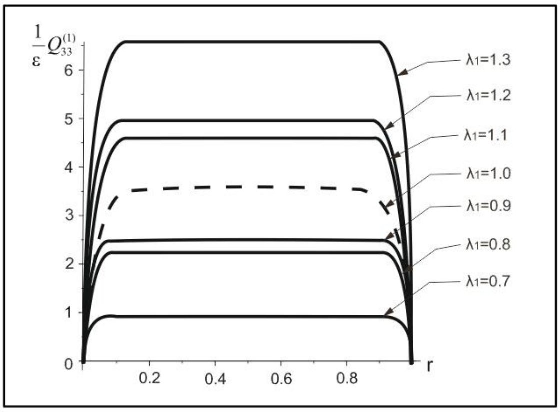

Fig. 2. Normal contact voltages. Harmonic potential

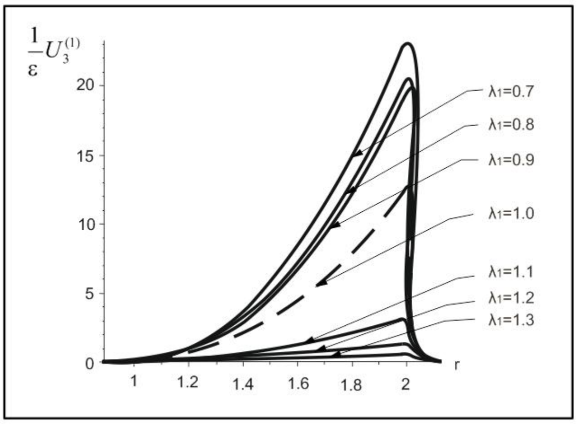

Fig. 3. Normal contact movements. Harmonic potential

On Fig. 2, the origin along the r axis corresponds to the value.

The quantitative characteristics of the influence of the initial (residual) stresses (in percent) with respect to the half-space and the ring stamp without initial stresses are presented in Table 1.

Table 1. Influence of initial stresses on the contact interaction of an elastic half-space and an elastic ring stamp (potential of a harmonic type).

σ_{33} / σ_0	Decrease, %				Increase, %		
λ_1 \ r	0.7	0.8	0.9	1.0	1.1	1.2	1.3
0.1	72.6	25.5	33.1	0	37.4	48.3	96.9
0.2	74.3	30.1	37.4	0	28.6	38.7	84.1
0.3	74.4	30.3	37.7	0	28.0	38.1	83.4
0.4	74.4	30.4	37.7	0	28.0	38.1	83.3
0.5	74.4	30.4	37.7	0	28.0	38.1	83.3
0.6	74.4	30.4	37.7	0	28.0	38.1	83.3
0.7	74.4	30.3	37.7	0	28.0	38.1	83.4
0.8	74.3	30.1	37.4	0	28.6	38.7	84.1

0.9	72.6	25.5	33.1	0	37.4	48.3	96.9

From Table 1, we can see that the initial (residual) stresses under compression lead to a decrease in the stress force in the elastic ring stamp and half-space, and when stretching, they increase.

Table 2. Numerical values of the force P/ε.

Потенциал гармонического типа	λ_1	0,7	0,8	0,9	1	1,1	1,2
	P/ε	0,3458	0,4128	0,4401	0,4521	0,4567	0,4576

Table 2 also shows the ratio of the numerical values of the force P, which acts on the upper end of the resilient stamp, for given initial (residual) stresses with a case without initial stresses. The case without initial (residual) stresses is highlighted in table 2 in bold type.

4.2. Analysis of the regularity of the influence of initial (residual) stresses on the distribution of contact characteristics

To determine the effect of initial stresses on contact voltages and the displacement (z_1=0) of an elastic half-space and an annular stamp with initial (residual) stresses, it is convenient to use the functions introduced earlier by A.N. Guzem (1991). Taking into account [7] and (6) - (10), for a stamp and a half-space with initial (residual) stresses, we introduce potentials in a system of cylindrical coordinates (r, θ, z_1).

For equal roots of the defining equation (3):

$$\varphi_1(r,z_1)=(1+m_1)^{-1}\cdot f(r,z_1),\quad \varphi_2(r,z_1)=-(1+m_2)^{-1}\cdot f(r,z_1) \qquad (19)$$

Besides:

$$\varphi_1=-\frac{\partial\chi_1}{v_1\partial z_1},\quad \varphi_2=-\chi_2 .$$

Taking into account (19) at the contact boundary at z_1 = 0 for a stamp and a half-space with initial (residual) stresses, we have the relation:

$$\mathrm{U}_3^{(1)}(r,z_1)=(1+2m_1-m_2)(v_1(1+m_1)(1+m_2))^{-1}\cdot\partial f(r,z_1)/\partial z_1,\quad Q_{33}^{(1)}(r,z_1)=C_{44}(l_1-l_2)\cdot\partial^2 f(r,z_1)/\partial z_1^2$$

$$\mathrm{U}_r^{(1)}(r,z_1)=(m_2-m_1)((1+m_1)(1+m_2))^{-1}\cdot\partial f(r,z_1)/\partial r . \qquad (20)$$

Substituting (20), the second and third boundary conditions (7) and (8) for z_1 = 0, we obtain

$$\partial^2 f/\partial z_i^2=\theta_9^{-1}\sigma(r),\ \text{при } 0<r<R,\ z_i=0;\quad \partial f/\partial z_i=0,\ \text{при } R<r<\infty,\ z_i=0, \qquad (21)$$

$$\theta_9=\begin{cases}C_{44}(l_2-l_1),\ при\ n_1=n_2;\\ C_{44}(l_2v_2-l_1v_1),\ при\ n_1\neq n_2.\end{cases}$$

where σ(r) is the right-hand side of the second boundary condition (7),

The mixed axisymmetric problem (21), (3) for $z_1\leq 0$ in the case of the harmonic potential f coincides with the analogous problem for the harmonic potential in the classical formulation of the linear theory of elasticity [11], if substituting $\theta_9=0,5\mu$

Therefore, for a potential of a harmonic type, we obtain a solution in the form of a Fourier-Bessel integral [7]:

$$f(r,z_1)=\int_0^\infty \eta^{-1}\Phi^*(\eta)\,\mathrm{J}_0(r\eta)\,\mathrm{e}^{-\eta z_i}\,dz,\quad \Phi^*(\eta)=2(\pi\theta_9)^{-1}\int_0^R \sin(\eta t)\left(\int_0^t y\sigma(y)(t^2-y^2)^{-0.5}dy\right)dt$$

Comparing (20) with the corresponding expressions for an isotropic body without initial stresses [11], taking into account (21), we obtain expressions for displacements and stresses at z_1=0 in an elastic body with initial (residual) stresses in the form:

$$U_3(r,0)=\mathrm{k}\cdot U_3^0(r,0),\qquad U_r(r,0)=\mathrm{k}_r\cdot U_r^0(r,0),\qquad Q_{33}(r,0)=\mathrm{k}_s\cdot Q_{33}^0(r,0), \qquad (22)$$

where $U_3(r,0)$, $U_r(r,0)$, $Q_{33}(r,0)$ – displacement and stress under the stamp, which presses on an elastic half-space with initial (residual) stresses;

$U_3^0(r,0)$, $U_r^0(r,0)$, $Q_{33}^0(r,0)$ – displacement and stress under the stamp, which presses on an elastic half-space without initial (residual) stresses;

k, kr, ks are coefficients that reflect the influence of initial (residual) stresses on the contact interaction of elastic bodies.

As a result, we obtain for equal roots (3):

$$k = 2\mu(\mu+\lambda)(1+2m_1-m_2)(C_{44}\, v_1(2\mu+\lambda)(l_1-l_2)(1+m_1)(1+m_2))^{-1},$$
$$k_r = 2(\mu+\lambda)(m_1-m_2)(C_{44}(l_1-l_2)(1+m_1)(1+m_2))^{-1},\ k_s = C_{44}(l_1-l_2)(2\mu)^{-1} \quad (23)$$

Consider the numerical representation of the study for the potential of the harmonic type and the Bartenev-Khazanovich potential at E = 3.92, σ = 0.47.

Consequently, we obtain the conditions for the surface instability of the material. For potentials of a harmonious type and Bartenev-Khazanovich, they numerically correspond to values of the initial elongation coefficient λ1 presented in Tables 3-5 in the first two rows of the first column, respectively.

Table 3. Numerical dependence of k on the extension parameter λ_1

k / λ_1	Potential of the Bartenev-Khazanovich	Potential of the harmonic type
0.5951417003	-	∞
0.6933612743	∞	1.539602504
0.7	19.79133718	1.506070972
0.8	1.708756425	1.244628239
0.9	1.165274009	1.116645537
1	1	1
1.1	0.9328196304	0.8532624739
1.2	0.9048418542	0.6305659580
1.3	0.8960706168	0.2329799840

The dependence of the change in the coefficients k, kr, ks (23) from the equations (22), which determine the influence of the initial (residual) stresses on the contact stresses and the displacement of the elastic stamp and half-space, are presented in Tables 3-5 and Figures 4-6. It is seen that as the coefficient of elongation λ_1 approaches the values of the surface instability of the material, the displacements increase indefinitely, and the stresses approach zero.

Table 4. Numerical dependence of kr on the extension parameter λ1

k_r / λ_1	Potential of the Bartenev-Khazanovich	Potential of the harmonic type
0.5951417003	-	∞
0.6933612743	∞	23.45290413
0.7	12.10103448	21.75675679
0.8	1.582686567	9.011857735
0.9	1.184928391	4.003984057
1	1	1
1.1	0.8661844303	-1.190858047
1.2	0.7494455067	-2.971887550
1.3	0.6381810052	-4.517518655

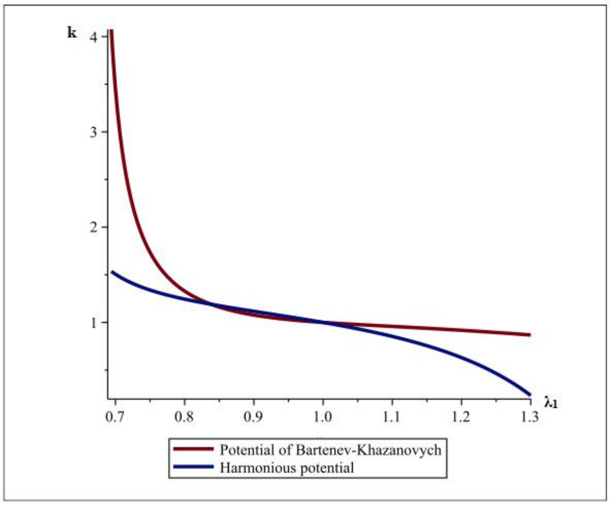

Figure 4. Graphical dependence of k on λ_1

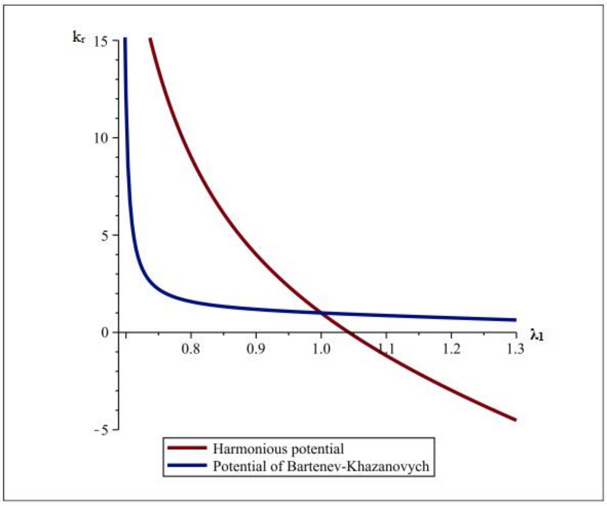

Figure 5. Graphical dependence of k_r on λ_1

From Tables 3 through 5 and Figures 4-6, it can be seen that the initial (residual) stresses during compression lead to a decrease in the stress force in the ring stamp and the base (half-space), and in tension, to their increase. In the case of displacements, everything is reversed.

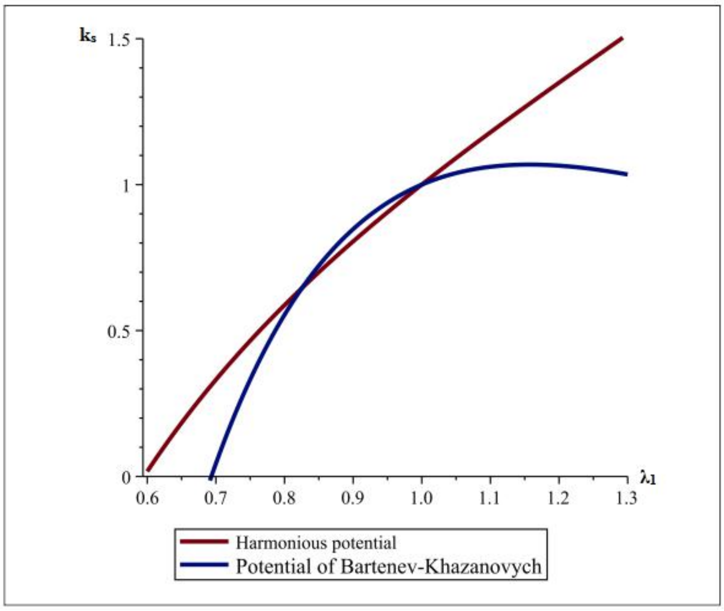

Figure 6. Graphical dependence of k_s on λ_1

Table 5. Numerical dependence of ks on the extension parameter λ1

k_s / λ_1	Potential of the Bartenev-Khazanovich	Potential of the harmonic type
0.5951417003	-	0
0.6933612743	0	0.3127940248
0.7	44.38413794	4.790678243
0.8	2.610746268	2.322267136
0.9	1.359713564	1.453924679
1	1	1
1.1	0.884737721	0.714235222
1.2	0.877782027	0.513225733
1.3	0.932724021	0.360898204

Thus, when the initial (residual) stresses approach the values of the surface instability of the material, effects of a "resonant" nature appear. Namely, a mechanical effect has been revealed, which is confirmed by earlier studies by Academician A.N. Guzy (for cracks) [10] and prof. V.B. Rudnitskii (for a finite stamp of arbitrary cross section without initial stresses with a half-space with initial (residual) stresses) [7].

4.3. Numerical results

On the basis of numerical analysis, it can be seen that the maximum values of contact displacements (Fig. 3) are achieved at points slightly closer to the outer side of the elastic ring stamp in the contact area.

In addition, it follows from numerical studies that, under constant external loading, a change in the elastic potential leads to a change in the character of the distribution of contact stresses and displacements under an elastic ring stamp. And the influence of the initial (residual) stresses on the stress-strain state of the elastic half-space, into which the elastic ring stamp is pressed, is as follows:

4.3.1. The initial (residual) stresses in the half-space and the stamp are reduced in the case of compression ($\lambda_1 <1$) to a decrease in stress, and in the case of tension ($\lambda_1> 1$) - to increase (Fig. 2, Table 1);

4.3.2. In the case of displacements (Fig. 3) - on the contrary. With compression ($\lambda_1 <1$), the initial (residual) stresses in the half-space and stamp cause an increase in displacements in absolute value, and in the case of stretching ($\lambda_1> 1$) - to reduce them. And for contact movements unsafe are the initial (residual) stresses in the case of compression, and for contact stresses - in the case of stretching;

4.3.3. In the absence of initial (residual) stresses ($\lambda_1=1$), the results obtained coincide with the classical ones [27].

4.3.4. It also follows from Table 2 that for a constant value of the end face of stamp ε for a potential of a harmonic type, the presence of initial (residual) stresses leads to the following:

4.3.4.1. In the case of compression ($\lambda_1 <1$) - the resultant load P decreases;

4.3.4.2. In the case of stretching ($\lambda_1>1$) - the resultant load P increases.

4.3.5. It is found that at $\lambda_1 \rightarrow \lambda_{kr}$ the influence of initial (residual) stresses in the elastic basis (half-space) is rapidly decreasing, and in the elastic ring stamp is approaching zero. That is, as in [7], a mechanical effect is found. And it consists in the fact that the elastic ring stamp is in a state of elastic equilibrium and has practically no effect on the substrate (half-space).

5. Discussion.

Analysis of the research results shows that the presence of a prestressed state in the contact interaction of an elastic ring stamp and an elastic half-space makes it possible to regulate the contact of stress and displacement when calculating the design and details of the mechanisms for strength.

Therefore, the practical significance of the findings of the study is that:

1. This research is aimed to solving the spatial axisymmetric static problem of the pressure of an elastic ring stamp on an elastic half-space with allowance for the initial stressed state. The results of the investigations made it possible to formulate the characteristic ratios for the potentials of an arbitrary structure for the components of the stress-strain state in the contact zone;

2. Analytical relations are obtained the reflect the influence of initial stresses on the law of distribution of contact stresses and displacements;

3. The principle of solution proposed in the article can be used to study various isotropic, transversal-isotropic or composite materials in the design of process equipment, columns of buildings and other structures.

6. Conclusion.

In this paper we present and solve a new static contact problem on the pressure of an elastic ring stamp with initial (residual) stresses on an elastic half-space (base) with initial (residual) stresses without allowance for friction forces.

The value of the studies carried out is that taking into account the influence of the initial (residual) stresses in the bodies on the law of distribution of the contact characteristics of elastic bodies at the points of their interaction can allow us to take into account, more effectively, the wear resistance of materials by properly estimating their strength reserves. Also, it can sufficiently reduce their material consumption, while retaining the necessary functional characteristics of materials.

Taking into account the obtained results, as well as previous studies on the contact interaction of elastic bodies [3, 6, 7, 10, 13, 14, 17-20, 23], the following conclusions can be set:

1 The initial (residual) stresses act in high-elastic materials in comparison with more rigid materials, but qualitatively, their influence persists;

2. The situation is unsafe when the initial (residual) stresses approach the values of the surface instability of the material, because the contact characteristics sharply change their values.

Consequently, the observed effect of the initial (residual) stresses is significant and must be taken into account when calculating the strength in structural details.

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9. Appendix.

In this paper, within the framework of the linearized theory of elasticity, the general relations of the stress-strain state of the contacting elastic bodies are written out using the reference method, which is given in the monographs of Academician A.N. Guzia [6, 7, 13]. The construction of analytical solutions for an elastic ring stamp and a half-space with initial (residual) stresses occurred using methods of separation of variables (the Fourier method) and Hankel integral transforms, respectively. We also used a method solving paired integral equations with their subsequent reduction to an integral equation of the Fredholm type of the second kind. In the article, the problem reduces to solve an infinite system of linear algebraic equations, which is solved by the reduction method and software tools (Maple 17).

Also, given that the study deals with elastic bodies, the main property of which is the recurrence of the processes occurring in them. We confine ourselves to the consideration of hyperelastic bodies. Here we have in mind bodies for which there exists an elastic potential Φ.

We assume that the potential Φ is a twice continuously differentiable function of the algebraic invariants of the Green's deformation tensor $\dot{\varepsilon}$ [7].

$$\Phi = \Phi(A_1, A_2, A_3), \tag{24}$$

$$A_1=\varepsilon_1+ \varepsilon_2+ \varepsilon_3, \quad A_2=\varepsilon_1^2+ \varepsilon_2^2+ \varepsilon_3^2, \quad A_3=\varepsilon_1^3+ \varepsilon_2^3+ \varepsilon_3^3, \tag{25}$$

where ε_i (i = 1,2,3) are the main values of the tensor $\dot{\varepsilon}$ of the Green's deformations. It means that the directions of the coordinate lines coincide with the principal directions of the tensor.

The potential (24) is convenient for compressible isotropic bodies. For incompressible isotropic bodies invariants of the form

$$I_1 = 3 + 2A_1, \quad I_2 = 3 + 4A_1 + 2(A_1^2 - A_2), \quad I_3 = 1 + 2A_1 + 2(A_1^2 - A_2) + 4/3(2A_3 - 3A_2A_1 + A_1^3)$$

Moreover, in the case of an incompressible body [7].

At researches sometimes use and the following system of invariants:

$$\tilde{s}_1=(\lambda_1-1)+(\lambda_2-1)+(\lambda_3-1).\quad \tilde{s}_2=(\lambda_1-1)^2+(\lambda_2-1)^2+(\lambda_3-1)^2.$$

$$\tilde{s}_3=(\lambda_1-1)^3+(\lambda_2-1)^3+(\lambda_3-1)^3 \tag{26}$$

This system of invariants can be applied to the nonlinear theory of elasticity of small deformations [7]. In this case, with a certain accuracy (26), will have the form

$$\tilde{s}_1 \equiv A_1.\quad \tilde{s}_2 \equiv A_2.\quad \tilde{s}_3 \equiv A_3.$$

Besides:

$$\varepsilon_i = 0,5(\lambda_i^2-1),\quad \delta_i=\lambda_i-1,\quad (i=\overline{1,3})$$

where $\delta_i -$ the relative coefficients of elongation.

We note that the form of the potential (24) does not in any way reduce the generalization of the formulation of the problem, since any system of invariants of the Green's deformation tensor can be expressed in terms of (25) [7]. In addition, it was shown in [7, 8, 13] that different systems of invariants of strain tensors (Almance and others) can be expressed in terms of (25).

For example, for an incompressible isotropic body in the rather general case [7], the elastic potential (24) can be represented by a series:

$$\Phi=\sum_{i,j} c_{i,j}\,(I_1-3)^i\,(I_1-3)^j$$

And for compressible isotropic bodies:

$$\Phi=\sum_{i,j,k} c_{i,j,k}\;\tilde{s}_1^i\tilde{s}_2^j\tilde{s}_3^k \tag{27}$$

If in (27) we confine ourselves to a quadratic approximation, we will accordingly obtain a potential of a harmonic type [28].

$$\Phi=\frac{1}{2}\lambda\,\tilde{s}_1+\mu\,\tilde{s}_2 \tag{28}$$

The simplest potential for incompressible isotropic bodies is the Bartenev-Khazanovich potential [6, 7, 13], in which invariants (26) are used:

$$\Phi=2\mu\,\tilde{s}_1$$

More complex structures of elastic potentials are partially presented in [7].

We represent the values of a number of quantities that relate to the potential of the harmonic type (28):

$$A_{i\beta}=\frac{\lambda}{\lambda_i\lambda_\beta}+\frac{\delta_{i\beta}(2\mu-\lambda(\lambda_1+\lambda_2+\lambda_3-3))}{\lambda_i^3},\qquad \mu_{ij}=\frac{2\mu-\lambda(\lambda_1+\lambda_2+\lambda_3-3)}{\lambda_i\lambda_j(\lambda_i+\lambda_j)},$$

$$S_0^{\beta\beta}=\lambda_\beta^{-1}\left[\lambda(\lambda_1+\lambda_2+\lambda_3-3)+2\mu(\lambda_\beta-1)\right]$$

$$n_1=n_2=\frac{\lambda_3^2}{\lambda_1^2},\quad \mu=\frac{E}{2(1+\sigma)},\quad \lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)},\quad \nu_1=\nu_2=\frac{\lambda_3}{\lambda_1},$$

$$\lambda_1=\lambda_2,\quad \lambda_3=\frac{2\mu+\lambda(3-2\lambda_1)}{\lambda+2\mu},\quad S_0^{11}=S_0^{22}=\frac{2\mu}{\lambda_1}(\lambda_1-\lambda_3)$$

$$\omega'_{ij\alpha\beta}=\frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\omega_{ij\alpha\beta},\quad \omega'_{1111}=\frac{\lambda-2\mu}{\lambda_3},\quad \omega'_{3333}=\frac{\lambda_3}{\lambda_1^2}(\lambda+2\mu),\quad \omega'_{1122}=\frac{\lambda}{\lambda_3},$$

$$\omega'_{1133}=\frac{\lambda}{\lambda_1},\quad \omega'_{1221}=\frac{\mu(2\lambda_1-\lambda_3)}{\lambda_1\lambda_3},\quad \omega'_{3113}=\frac{2\mu\lambda_3}{\lambda_1(1+\lambda_3)},$$

$$\omega'_{1313} = \frac{2\mu\lambda_3}{\lambda_1(\lambda_1+\lambda_3)}, \quad \omega'_{1212} = \frac{\mu}{\lambda_1}, \quad \omega'_{1331} = \frac{2\mu\lambda_1}{\lambda_3(\lambda_1+\lambda_3)},$$

$$c_{44} = \frac{2\mu\lambda_3}{\lambda_1(\lambda_1+3)}, \quad m_1 = \frac{\lambda_3}{\lambda_1}, \quad m_2 = \frac{\lambda\lambda_1+(\lambda-2\mu)\lambda_3}{\lambda\lambda_1+(\lambda+2\mu)\lambda_3},$$

$$l_1 = \frac{\lambda_1}{\lambda_3}, \quad l_2 = \frac{\lambda-2\mu}{2\lambda} - \frac{(\lambda+4\mu)l_1}{\lambda_1}, \quad s = \frac{2\mu+3\lambda-(3\lambda+4\mu)\lambda\lambda_1}{2\mu+3\lambda-\lambda\lambda_1},$$

$$s_0 = s_3 = \frac{\lambda\lambda_1}{2\mu+3\lambda-\lambda\lambda_1}, \quad s_1 = \frac{-4\mu\lambda_1}{(2\mu+\lambda)\lambda_3+\lambda\lambda_1}, \quad s_2 = \frac{(\lambda\lambda_1+(\lambda-2\mu)\lambda_3)\lambda_1}{(\lambda\lambda_1+(\lambda+2\mu)\lambda_3)\lambda_3}.$$

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