Contents

Chapter 1
Contact Problems for Cylindrical Stamps and Elastic Bodies with Initial (Residual) Stresses

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Abstract The chapter is devoted to contact problems of a cylindrical punch with prestressed bodies without consideration of friction. It highlights the problems concerning rigid or elastic ring-shaped punches on a half-space with initial stresses, problems for prestressed cylindrical punch and elastic layer, and the problem with two prestressed half-spaces with elastic cylindrical punches with initial stresses. The results are expressed in the general form for the theory of large initial deformations and two options for the theory of small initial deformations under the arbitrary form of an elastic potential. Many fundamental results are used to study the problem, such as Hankel's transformations, dual integral equations, orthogonal polynomials, and other contact mechanics methods of linearized elasticity theory.

1.1 Introduction

Studying the problems of contact mechanics, which is the primary purpose of this chapter, is a fundamental issue since contact interaction is one of the most common ways of transmitting external loads in practice. Contact mechanics allows us to find die-pressure distribution, study its concentration, and develop ways to reduce it. The significance of this problem has no doubts from the point of view of the development of fundamental achievements in contact mechanics as from the point of view of the applied branches of modern technology. Furthermore, contact mechanics theory is critical in mechanical engineering since the contact of structural members with each other carries out the force transfer in machines and columns of buildings. Also, similar problems can occur when calculating the critical characteristics of foundations of building columns, chimneys, cooling towers, water towers, and other high-rise structures for wind load or load from their weight.

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The circle of problems about the contact of elastic, viscoelastic, and plastic bodies without initial stresses is quite broad [1]. Nevertheless, modern engineering practice demands have presented some new problems to researchers. They require the use of more complex continuous medium models. These are problems with complicated physical and mechanical properties. Models of contact interaction must consider the following factors: heat generation, the effect of friction, stiffness, surface properties of the material, and wear resistance of the surface.

Another fundamental problem of contact mechanics is the consideration of initial stresses, which need to be studied more intensely. Various factors, such as technological operations, manufacturing processes, and assembly of structures, cause initial stresses. When applying geostatic and geodynamic forces, the initial stresses arise in the earth's crust. We consider initial stresses when solving problems about the deformation of soils (especially frozen ones). They arise due to technological processes in the creation of composite materials. Initial stresses are present in the blood vessels of living organisms. Internal residual stresses may exist in elastoplastic bodies after the removal of the load. Sometimes it is appropriate to deliberately create initial stresses (residual and technological) to compensate for those that arise in structural members. Such an approach increases the strength characteristics of the structure. Of particular interest is the study of contact problems for prestressed bodies due to the manufacturing of new artificial materials that can withstand large initial deformations. Considering the initial stresses in the approbation of critical structural members makes it possible to use strength resources more effectively, significantly reducing material consumption.

Quite often, in order to increase the strength of the structure, there is a need to strengthen some of its load-bearing elements with elastic fasteners (stringers). The results of research that accounted for initial stresses in the structure were carried out in [2, 3]. As in the mentioned papers, the current study is characterized by all considered punches being elastic, and the bases are prestressed.

Two approaches have historically been developed while studying problems of contact interaction of bodies with initial stresses. The first one is related to studying bodies with a specific form of elastic potential. The paper [4] became pioneering in this field of research. It deals with the problem of a coin-shaped crack in the case of an incompressible elastic body with initial stresses for the Treloar potential (a body of the neo-Hookean type). Studies related to this approach are reviewed in [5, 6] for other potentials.

The second approach [7, 8, 9] is developed in parallel with the first one. It is related to the study of problems formulated for elastic bodies with initial stresses with an arbitrary structure of elastic potential. In the mentioned papers, the problems are solved in a general way for compressible and incompressible materials. For this purpose, the theories of large (finite) initial strains and two versions of the theory of small initial strains for equal and unequal roots of the characteristic equation are used [10, 11]. All the results presented in this chapter are obtained within the framework of the second approach. The authors believe it has several advantages compared to the first approach.

Thus, the same problem (contact or crack problem) for prestressed bodies was considered by some authors, for example, for the Treloar potential and by other authors for the Mooney potential, that is, for a specific form of elastic potential. This chapter presents the research results in a single general form for compressible and incompressible prestressed bodies for an arbitrary structure of the elastic potential. Specific elastic potentials were used only at the final research stage (obtaining numerical results).

Contact problems for rigid and elastic punches within the framework of the second approach in Ukraine were solved in the works of Guz, Rudnytskyi, Hryhorenko, Ramskyi, Dikhtyaruk, Glukhov, Prymachenko, Matnyak, Babych, and Yaretska [10, 11, 12, 13, 14, 15, 16, 17]. In the current chapter, using the relations of the linearized theory of elasticity, the main results of research on contact interaction are presented, namely: the 3D problem on a prestressed cylindrical punch and an elastic layer with initial stresses [13, 16]; problems dealing with the pressure of two prestressed half-spaces on an elastic cylindrical punch with initial stresses [18]; problems on a rigid ring-shaped punch with a half-space with initial stresses [14]; problems dealing with the pressure of a prestressed elastic ring-shaped punch with a flat base on a half-space with initial stresses [15]. The results of the study are expressed in a general form without taking into account the friction for compressible and incompressible bodies. We obtained the solutions within the theory of large (finite) initial deformations and two theories of small initial deformations with an arbitrary form of the elastic potential (within the framework of the second approach).

As the initial stresses can not be taken into account in the linear elastic mechanics of materials, the general nonlinear theory of elasticity [7, 9, 19] can be used. However, it is challenging to get a solution directly in this case. Therefore, with sufficiently large initial stresses, it is better to use its linearized option [10, 11, 16].

Thus, let us assume that the following fundamental conditions of linearized elasticity theory hold [11]:

1. Contact interaction of an elastic finite cylindrical or ring-shaped punch with initial stresses (or without it) with a prestressed elastic body (layer or half space) occurs after appearing of the initial stress state.
2. Additional (relative to the initial state) external loading causes much less stress-strain disturbance in prestressed bodies compared to the corresponding values of the initial stress state.
3. The initial stress-strain state in the area of contact interaction can be approximately considered homogeneous.
4. The solution of linearized problems of the elasticity theory of the contact interaction of prestressed bodies and punches (cylindrical or circular) is unique [11]. Thus, condition (2.23) in [11] is satisfied.

The second condition can be violated in the points where boundary conditions change [11]; at these points, the contact stresses are unbounded. A detailed discussion of this phenomenon in the theory of contact problems of the linear and linearized theory of elasticity is given in [1, 11]. Based on the results of mentioned papers, the following can be concluded. There are power singularities in the solutions of contact problems for elastic and rigid bodies. The contact stresses are $O(\rho^{1-\gamma})$, where ρ is the distance from a point to the point where boundary conditions change, γ is a parameter that is determined by a transcendental equation [1] and depends on the elastic constants of contacting bodies, as well as on the elastic potential. Thus, at the mentioned points, the stresses have no physical sense and do not affect determining the integral characteristics of the contact problems.

1.2 Main Relations

Let elastic solids interact with elastic or rigid punches. The surfaces outside the contact boundary are assumed to remain free from the influence of external forces, and displacements and stresses are continuous at the contact boundary.

We use the coordinates Oy_i $(i = 1, 2, 3)$, which are associated with the Lagrangian coordinates x_i $(i = 1, 2, 3)$ by the relations: $Oy_i = \lambda_i x_i$ $(i = 1, 2, 3)$, where λ_i $(i = 1, 2, 3)$ are the elongation factors determining displacements of the initial state, λ_i = const. The y_3-axis is normal to the contact boundary.

Consider elastic isotropic bodies (compressible or incompressible) with an arbitrary form of the elastic potential [11]. For orthotropic bodies, the elastic-equivalent directions are assumed to coincide with the coordinate axes in the deformed state y_i $(i = 1, 2, 3)$.

We assume that the initial stress state is homogeneous, the contact boundary of elastic bodies is in the plane y_3 = const, and the initial stresses act along the contact boundary [11, 16]

$$y_m = x_m + U_m^0, \quad U_m^0 = \delta_{mi}(\lambda_m - 1)\lambda_i^{-1} y_i \quad (i = 1, 2, 3),$$

where δ_{mi} is the Kronecker symbol.

Then, for compressible bodies, the basic equation in terms of displacements [11, 16] is the following

$$L'_{m\alpha} U_\alpha = 0, \quad L'_{m\alpha} = \frac{\omega'_{ij\alpha\beta}\partial^2}{\partial y_i \partial y_{\beta_i}} \quad (i, m, \alpha, \beta = 1, 2, 3); \tag{1.1}$$

for non-rigid bodies, the incompressibility condition is satisfied

$$\begin{gathered} L'_{m\alpha} U_\alpha + \frac{q'_{\alpha m}\partial p'}{\partial y_\alpha} = 0, \quad L'_{m\alpha} = \frac{\kappa'_{ij\alpha\beta}\partial^2}{\partial y_i \partial y_{\beta_i}}, \\ q'_{ij}\frac{\partial U_j}{\partial y_i} = 0, \quad q'_{ij} = \lambda_i q_{ij} \quad (i, j, m, \alpha, \beta = 1, 2, 3). \end{gathered} \tag{1.2}$$

Stress tensor components for compressible bodies at y_i = const $(i = 1, 2, 3)$ reeds

$$Q'_{ij} = \omega'_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta}, \quad \omega'_{ij\alpha\beta} = \frac{\lambda_i \lambda_\beta}{\lambda_1 \lambda_2 \lambda_3}\omega_{ij\alpha\beta},$$

and for non-compressible bodies

$$Q'_{ij} = \kappa'_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial y_\beta} + q'_{ij}p, \quad \kappa'_{ij\alpha\beta} = \frac{\lambda_i\lambda_\beta}{\lambda_1\lambda_2\lambda_3}\kappa_{ij\alpha\beta},$$

where $\omega'_{im\alpha\beta} = \omega'_{im\alpha\beta}\left(S_{11}{}^0,\ S_{22}{}^0,\ S_{33}{}^0\right)$ and $\kappa'_{im\alpha\beta} = \kappa'_{im\alpha\beta}\left(S_{11}{}^0,\ S_{22}{}^0,\ S_{33}{}^0\right)$ are the components of the fourth-order tensor of elasticity modules.

The initial stresses being homogeneous, the following condition takes place

$$S_0^{11} = S_0^{22} \neq 0, \quad S_0^{33} = 0, \quad \lambda_1 = \lambda_2 \neq \lambda_3. \tag{1.3}$$

Taking into account (1.3), the solution of (1.1) and (1.2) can be represented using function $\tilde{\chi}$ that, in cylindrical coordinates (r, θ, y_3), satisfies the characteristic equation

$$\left(\Delta_1 + \xi_2'^2\frac{\partial^2}{\partial y_3^2}\right)\left(\Delta_1 + \xi_3'^2\frac{\partial^2}{\partial y_3^2}\right)\tilde{\chi} = 0, \tag{1.4}$$

where $\Delta_1 = \partial^2/\partial r^2 + \partial/\partial r$.

We take into account the uniqueness of the solution of the linearized theory of elasticity for compressible and incompressible bodies and present two possible variants of the general solution of (1.4):

- the case of equal roots ($\xi_2'^2 = \xi_3'^2$):

$$\tilde{\chi} = \tilde{\chi}_1 + y_3\tilde{\chi}_2, \quad \left(\Delta_1 + \xi_2'^2\frac{\partial^2}{\partial y_3^2}\right)\tilde{\chi}_1 = 0, \quad \left(\Delta_1 + \xi_2'^2\frac{\partial^2}{\partial y_3^2}\right)\tilde{\chi}_2 = 0; \tag{1.5}$$

- case of unequal roots ($\xi_2'^2 \neq \xi_3'^2$):

$$\tilde{\chi} = \tilde{\chi}_1 + \tilde{\chi}_2, \quad \left(\Delta_1 + \xi_2'^2\frac{\partial^2}{\partial y_3^2}\right)\tilde{\chi}_1 = 0, \quad \left(\Delta_1 + \xi_3'^2\frac{\partial^2}{\partial y_3^2}\right)\tilde{\chi}_2 = 0. \tag{1.6}$$

In circular cylindrical coordinates (r, θ, z_i) ($z_i = \upsilon_i^{-1}y_3$, $\upsilon_i = \sqrt{n_i}$ $(i = 1, 2)$, $n_1 = \xi_2{}'^2$ and $n_2 = \xi_3{}'^2$) we obtained the solutions for finite cylindrical (circular) punches with initial stresses using variable separation methods (Fourier method) in the form of infinite system for constants. The stress-strain state in elastic bodies with initial stresses for cases (1.5) and (1.6) is determined in terms of harmonic functions in the form of Hankel integrals.

Further, we consider the solved problems in detail.

1.3 Spatial Contact Problem for Prestressed Cylindrical Punch and Elastic Layer With Initial (Residual) Stresses

The section presents the problem statement, boundary conditions, method of solution, and numerical results.

1.3.1 Problem Statement and Boundary Conditions

Let the finite elastic cylindrical punch (with the radius of the base R and height H) with initial stresses be pressed into the elastic layer under the action of force P (Fig. 1.1). We use the following notations in Fig. 1.1: h_1 is the thickness of the layer in the initial stress state, which is related to the thickness h_2 in the undeformed state by the ratio $\lambda_3 = h_2/h_1$. We assume that P is applied only to the free end of the elastic punch. All the points of the end of the punch move in the direction of the symmetry axis y_3 by the same value ε. In the case of a prestressed layer, three types of contact interaction are considered: 1) the layer is located on a rigid foundation without friction; 2) the layer with initial stresses is rigidly fixed on an undeformed foundation; 3a) the layer with the initial stresses lays without friction on the foundation with the initial stresses; 3b) the layer with initial stresses lays without friction on a foundation without initial stresses.

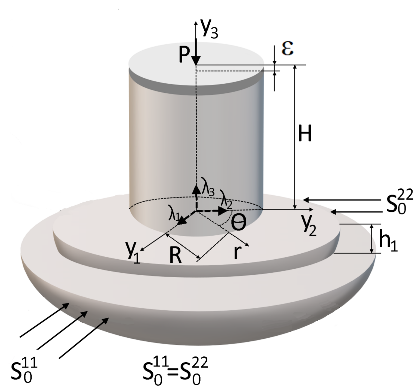

Figure 1.1 Prestressed cylindrical punch on the elastic layer with initial stresses.

In a cylindrical coordinate system (r, θ, z_i), the following boundary conditions correspond to the problem statement:

1. At the end of the elastic punch $z_i = H\nu_i^{-1}$:

$$u_3'^{(1)} = -\varepsilon, \quad Q_{3r}'^{(1)} = 0 \quad (0 \leqslant r \leqslant R). \tag{1.7}$$

2. On the edge of the elastic layer in the contact area:

$$u_3'^{(1)} = u_3'^{(2)}, \quad Q_{33}'^{(1)} = Q_{33}'^{(2)}, \quad Q_{3r}'^{(1)} = Q_{3r}'^{(2)} = 0$$
$$(z_i = 0, \quad i = 1, 2; \quad 0 \leqslant r \leqslant R). \quad (1.8)$$

3. On the edge of the elastic layer outside the contact area:

$$Q_{33}'^{(2)} = 0, \quad Q_{3r}'^{(2)} = 0 \quad (z_i = 0, \quad i = 1, 2; \quad R \leqslant r < \infty). \quad (1.9)$$

4. On the side surface of the elastic punch:

$$Q_{rr}'^{(1)} = 0, \quad Q_{3r}'^{(1)} = 0 \quad (0 \leqslant z_i \leqslant Hv_i^{-1}; \quad r = R). \quad (1.10)$$

5. On the lower surface of the layer $z_i = -\lambda_3 h_2 v_i^{-1} = -Hv_i^{-1}$ ($i = 1, 2$):

   a. for a layer with initial stresses lying without friction on an undeformed foundation:
   $$u_3'^{(2)} = 0, \quad Q_{3r}'^{(2)} = 0 \quad (0 \leqslant r < \infty); \quad (1.11)$$
   b. for a layer with initial stresses rigidly fixed to an undeformed foundation:
   $$u_3'^{(2)} = 0, \quad u_r'^{(2)} = 0 \quad (0 \leqslant r < \infty); \quad (1.12)$$
   c. for a layer with initial stresses lying without friction on an elastic basis with initial stresses:
   $$u_3'^{(2)} = u_3'^{(3)}, \quad Q_{3r}'^{(2)} = Q_{3r}'^{(3)} = 0 \quad (0 \leqslant r < \infty). \quad (1.13)$$

1.3.2 Method of Solution

Solutions are found in the case of roots (1.5) and (1.6) of (1.4). For example, the solution in general form for a cylindrical punch reads

- for $n_1 = n_2$:

$$\tilde{\chi} = \varepsilon \Bigg\{ v_1 z_1 \left[(m_2 - 1)^{-1} + \chi_0 \left((1 - m_2)^{-1} - 2E(3H\theta_2)^{-1}(3r^2 - 2z_1^2) \right) \right]$$
$$+ R \sum_{k=1}^{\infty} \chi_k \Bigg[R(2\gamma_k)^{-1} b_1^{(k)} \left(H \left(1 + \frac{s_0(1 - I_0(v_1\gamma_k R))}{v_1\gamma_k R I_1(v_1\gamma_k R)} \right) + z_1 \right) I_0(\gamma_k z_1 v_1)$$
$$\times \sin(\gamma_k z_1 v_1) + J_0(\alpha_k r)\mu_k^{-1}(\tilde{S}_2(\alpha_k z_1) + z_1\tilde{S}_3(\alpha_k z_1)) \Bigg] \Bigg\};$$

- for $n_1 \neq n_2$:

$$\tilde{\chi} = \frac{\varepsilon}{2}\left\{\frac{1}{\theta_8}(r^2 - z_1^2 - z_2^2) - \chi_0\left[r^2\left(\frac{1}{\theta_8} + \frac{z_1+z_2}{2H\theta_6}\right) - \frac{z_1^2+z_2^2}{\theta_8} - \frac{z_1^3+z_2^3}{3H\theta_6}\right]\right\}$$
$$-\sum_{k=1}^{\infty}\left\{b_3^{(k)}\left[s_0\frac{I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)}I_0(\gamma_k v_1 r)\sin(\gamma_k z_1 v_1) + I_0(\gamma_k v_2 r)\sin(\gamma_k z_2 v_2)\right]\right.$$
$$\left.-J_0(\alpha_k r)\left[\tilde{S}_2(\alpha_k z_1) + \tilde{S}_3(\alpha_k z_2)\right]\right\}\chi_k,$$

where $J_v(x)$, $I_v(x)$ are Bessel functions of real and imaginary arguments; for $n_1 = n_2$, $\gamma_k = 2\pi k H^{-1}$; for $n_1 \neq n_2$, $\gamma_k = \pi(2k+1)H^{-1}$, $(k = 0, 1, 2, \ldots)$;

$$s_0 = (1+m_2)(1+m_1)^{-1}, \quad \theta_8 = m_1 n_1^{-1} + m_2 n_2^{-1}, \quad \theta_6 = m_1 v_1^{-3} + m_2 v_2^{-3},$$

$$b_3^{(k)} = 4\varepsilon R^2 J_0(\nu_k)\left[\frac{\tilde{c}_1 - \tilde{c}_0}{\nu_k^2 + (\gamma_k v_1 R)^2} - \frac{v_2}{v_1 s_0}\frac{\tilde{c}_2 - \tilde{c}_0}{\nu_k^2 + (\gamma_k v_2 R)^2}\right]$$
$$\times \frac{v_1 H \gamma_k^3 I_1(\gamma_k v_2 R)}{v_2 W_k(2) - v_1 s_0 W_k(1)}, \quad W(j) = \frac{(\tilde{c}_0 - \tilde{c}_j) I_0(\gamma_k v_j R)}{I_1(\gamma_k v_j R)} + \frac{1 - \tilde{c}_0}{\gamma_k v_j R},$$

$$m_1 = \begin{cases} (\omega'_{1111} n_1 - \omega'_{1331})(\omega'_{1133}\omega'_{1313})^{-1}, & \text{for compressible bodies} \\ \lambda_1 q_1 n_1 (\lambda_3 q_3)^{-1}, & \text{for non-compressible bodies,} \end{cases}$$

$$\tilde{c}_0 = \begin{cases} \omega'_{1111}\omega'^{-1}_{1122}, & \text{for compressible bodies} \\ \lambda_1 q_1 (\lambda_3 q_3)^{-1}(\kappa'_{1133} + \kappa'_{1313})\kappa'_{1122}, & \text{for non-compressible bodies,} \end{cases}$$

$$m_2 = \begin{cases} (\omega'_{1133} - \omega'_{1313})(\omega'_{1133} + \omega'_{1313})^{-1}, & \text{for compressible bodies,} \\ 1, & \text{for non-compressible bodies,} \end{cases}$$

$$\tilde{c}_i = \begin{cases} \lambda_3 \omega'_{1133} m_i \omega'^{-1}_{1122} n_i^{-1}, & \text{for compressible bodies;} \\ (\kappa'_{1133} m_i - \kappa'_{3113})\kappa'^{-1}_{1122} n_i^{-1} (i = 1, 2), & \text{for non-compressible bodies,} \end{cases}$$

$$b_1^{(k)} = \frac{J_0(\nu_k)\gamma_k t_{00}\left[t_{14}\sinh^2(\alpha_k H v_1^{-1})\cosh(\alpha_k H v_1^{-1}) + t_{11} t_{29}\right]}{(I_0(\alpha_k v_1 R) - 1)t_{33} + c_1(\cosh(\alpha_k H v_1^{-1}) - 1)t_{34} + \sinh(\alpha_k H v_1^{-1})t_{35}},$$

$$M^{(k)} = M_k N_k^{-1}, \quad E^{(k)} = -E_k N_k^{-1},$$

$$\tilde{S}_2(\alpha_k z_1) = R s_0 \mu_k^{-1}\cosh(\alpha_k z_1) + E^{(k)}\sinh(\alpha_k z_1),$$
$$\tilde{S}_3(\alpha_k z_2) = -\sinh(\alpha_k z_1) - M^{(k)}\cosh(\alpha_k z_1),$$

$$
\begin{aligned}
&t_{00} = \mu_k^2(\mu_k^2 + R^2 v_1^2 \gamma_k^2), \quad t_{01} = (1 + m_2 t_{23} + t_{10}\tilde{c}_1 s_0), \quad t_{02} = 2(H\tilde{c}_1\alpha_k - \tilde{c}_0 s_0),\\
&t_{03} = 2(m_1 + 1)(m_2 - 1), \quad t_{10} = \alpha_k H(m_1 + 1),\\
&t_{14} = \alpha_k H\tilde{c}_1\left(v_1(m_1 + 1)(m_2 - 1) + m_1(1 + m_2)\right), \quad t_{12} = \alpha_k H\tilde{c}_0 - \tilde{c}_1 s_0,\\
&t_{13} = v_1(\tilde{c}_2 - \tilde{c}_1 + \tilde{c}_0) + \alpha_k H\tilde{c}_1 - \tilde{c}_0 s_0, \quad t_{11} = v_1(1 - m_2) + s_0 m_1,\\
&t_{15} = v_1(\tilde{c}_0 - \tilde{c}_1 + \tilde{c}_2)t_{10} - \tilde{c}_1 s_0(m_2 + 1), \quad t_{23} = v_1(\tilde{c}_2 - \tilde{c}_1 + \tilde{c}_0) - \tilde{c}_0 s_0,\\
&t_{21} = H(H(3 + 2\tilde{c}_2 - 4\tilde{c}_1 + \tilde{c}_0) + 1 + 2\tilde{c}_1 + \tilde{c}_0), \quad t_{22} = 2v_1^4\gamma_k^2 R^2(m_2 - 1),\\
&t_{24} = v_1(1 + m_1)(\tilde{c}_2 - \tilde{c}_1 + \tilde{c}_0) + \tilde{c}_0(1 + m_2),\\
&\qquad J_1(\alpha_k R) = 0, \quad \alpha_k = \mu_k R^{-1},\\
&t_{27} = c_0\sinh(\alpha_k H v_1^{-1}) + c_1(1 - \cosh(\alpha_k H v_1^{-1})), \quad t_{26} = t_{12}\sinh(\alpha_k H v_1^{-1}) + t_{13},\\
&t_{28} = t_{10}\cosh(\alpha_k H v_1^{-1})t_{27} + c_1(1 + m_2)\sinh(\alpha_k H v_1^{-1})\left(1 - \cosh(\alpha_k H v_1^{-1})\right),\\
&t_{29} = (1 + m_1)\sinh^2(\alpha_k H v_1^{-1})t_{26} + \cosh(\alpha_k H v_1^{-1})t_{28},\\
&t_{30} = t_{12}\cosh(\alpha_k H v_1^{-1}) + c_1 s_0 + t_{23}\sinh(\alpha_k H v_1^{-1}),\\
&t_{31} = c_1 t_{10}\cosh(\alpha_k H v_1^{-1})(1 + \cosh(\alpha_k H v_1^{-1})),\\
&t_{32} = t_{31} + (1 + m_1)t_{30}\sinh(\alpha_k H v_1^{-1}), \quad t_{33} = t_{11}t_{21}t_{00}\sinh^2(\alpha_k H v_1^{-1}) + t_{22}t_{32},\\
&t_{34} = t_{10}\cosh(\alpha_k H v_1^{-1}) + (1 + m_2)\sinh(\alpha_k H v_1^{-1}),\\
&t_{35} = c_0 t_{10}\cosh(\alpha_k H v_1^{-1}) - t_{24}\sinh(\alpha_k H v_1^{-1}).
\end{aligned}
$$

Applying (1.7)–(1.13), the stress-strain state in the prestressed layer is defined for equal roots (1.5) (1.1) by

$$
\begin{aligned}
u_3^{(2)} &= \theta_3\left(\int_0^\infty \frac{F(\eta)}{\eta}J_0(\eta\rho)d\eta - \int_0^\infty \frac{F(\eta)}{\eta}G(\eta h)J_0(\eta\rho)d\eta\right),\\
Q_{33}^{(2)} &= \theta_1\int_0^\infty F(\eta)J_0(\eta\rho)d\eta, \quad Q_{3r}^{(2)} = 0,
\end{aligned}
\tag{1.14}
$$

where

$$
\theta_1 = C_{44}l_1(1 + m_1)\kappa, \quad h = \frac{h_1}{R}, \quad \theta_3 = \frac{m_1}{v_1}(s_1 - s_0), \quad s_1 = \frac{m_1 - 1}{m_1},
$$

$$
C_{44} = \begin{cases} \omega'_{1313}, & \text{for compressible bodies}\\ \kappa'_{1313}, & \text{for non-compressible bodies,}\end{cases}
$$

$$
m_1 = \begin{cases} \dfrac{\omega'_{1111}n_i - \omega'_{3113}}{\omega'_{1133} + \omega'_{1313}}, & \text{for compressible bodies}\\ \dfrac{\lambda_1 q_1}{\lambda_3 q_3}n_i, & \text{for non-compressible bodies,}\end{cases}
$$

$$I_i = \begin{cases} \dfrac{\omega'_{1331}}{\omega'_{1313}} + \dfrac{\omega'_{1313} - \omega'_{1331}}{\omega'_{1313}} \dfrac{\omega'_{1133} + \omega'_{1313}}{\omega'_{1111} n_i + \omega'_{1133}}, & \text{for compressible bodies} \\ \dfrac{\kappa'_{1331}}{\kappa'_{1313}} + \dfrac{\kappa'_{1313} - \kappa'_{1331}}{\kappa'_{1313}} \dfrac{\lambda_3 q_3}{\lambda_3 q_3 + \lambda_1 q_1 n_i}, & \text{for non-compressible bodies,} \end{cases}$$

In (1.14), we untroduces the notation $F(\eta) = \eta^3 B_2 R^{-3}(1-G(\eta))^{-1}$; the function $G(\eta)$ is determined by (1.7)–(1.13).

Next, we introduce variables χ_k $(k = 0, 1, 2, \dots)$, which define the stress state in the layer, punch, and foundation for equal (1.5) and unequal roots (1.6); these variables depend on the elastic potential.

From the conditions of continuity of stresses and displacements (1.8) at the contact boundary and outside it, we write down the dual integral equations with the unknown function $F(\eta)$:

$$\int_0^\infty F(\eta)\eta^{-1} J_0(\eta\rho)d\eta = f(\rho) \quad (\rho < 1),$$
$$\int_0^\infty F(\eta) J_0(\eta\rho)d\eta = 0 \quad (\rho > 1),$$

where, in the case of unequal roots, we have

$$f(\rho) = \frac{\varepsilon}{\theta_3}\left(\chi_0 - 1 - \theta_4 \sum_{k=1}^{\infty} \chi_k J_0(\mu_k\rho) + \frac{\theta_3}{\varepsilon}\int_0^\infty \frac{\eta}{F(\eta)} G(\eta h) J_1(\eta\rho)d\eta\right),$$

$\theta_4 = n_1^{-1}\left(v_1(m_2 - 1) - m_1 s_0\right)$.

Using the inversion formula [11], we obtain integral Fredholm equations of the second kind with the unknown function $F(\eta)$ determined by (1.5):

$$\begin{aligned} \frac{F(\eta)}{\eta} = &-\frac{2\varepsilon}{\pi\theta_3}\left((1-\chi_0)\Psi_0(\eta,0) - 2(m_2-1)\frac{R^2}{\theta_2}\chi_0\Psi_1(\eta,0)\right) \\ &- \frac{2\varepsilon}{\pi\theta_3}\left(\theta_4\sum_{k=1}^{\infty}\chi_k\Psi_0(\eta,\mu_k) + \frac{m_2-1}{2}R^2\sum_{k=1}^{\infty} b_1^{(k)}\chi_k\Psi_0(\eta, i\gamma_k v_1 R)\right) \\ &+ \frac{2}{\pi}\int_0^\infty \frac{F(u)}{u} G(uh)\Psi_0(\eta,u)du \end{aligned} \quad (1.15)$$

for equal roots (1.5), and

$$\begin{aligned} \frac{F(\eta)}{\eta} = &\frac{2\varepsilon}{\pi\theta_3}\left((\chi_0 - 1)\Psi_0(\eta,0) - \theta_4\sum_{k=1}^{\infty}\chi_k\Psi_0(\eta,\mu_k)\right) \\ &+ \frac{2}{\pi}\int_0^\infty \frac{F(u)}{u} G(uh)\Psi_0(\eta,u)du \end{aligned} \quad (1.16)$$

for unequal roots (1.6). In (1.15) and (1.16)

$$\Psi_n(x, y) = \int_0^1 t^n \cos(xt)\cos(y)t\,dt.$$

We use the method of successive approximations to find the solution of (1.15) and (1.16) in the form

$$F(\eta) = \sum_{k=1}^{\infty} F^{(k)}(\eta). \tag{1.17}$$

The convergence of the method of successive approximations is studied based on the principle of compressed mappings. Note that the process of successive approximations (1.17) is convergent for $\lambda_1 > \lambda_{kp}$. This method was used when solving equations (1.14) and (1.15) under the condition

$$h > v_1\sqrt{D_1(2\pi)^{-1}}, \quad D_n = \int_0^{\infty} t^n G(t)\,dt.$$

In the case of harmonic potential, the minimum layer thickness values h are given in [16]. In the case of the absence of initial stresses in the cylinder, the thickness of the layer is given for comparison. It was shown that the initial stresses affect the implementation of the method of successive approximations.

Using the boundary conditions (1.7)–(1.13) and the orthogonality of the Bessel functions, we obtain the infinite quasi-regular system of linear equations

$$\vartheta_k \chi_k + \sum_{n=0}^{\infty} \vartheta_{kn}\chi_n = \bar{\omega}_k \qquad (k = 0, 1, 2, \ldots). \tag{1.18}$$

The quasi-regularity of (1.18) can be established using asymptotic representations for the Bessel functions, and μ_k and boundedness of integrals $\Psi(\mu_k, \mu_n)$ for $\lambda_1 > \lambda_{kp}$ [16]. Thus, the problem is reduced to determining the constants χ_k $(k = 0, 1, 2, \ldots)$, by which the characteristics of the stress-strain state of the elastic layer, punch, and foundation with initial stresses are expressed. The relation between the punch base displacements and the load P for equal (1.5) and unequal roots (1.6) of (1.4) are the following

$$P = 8\pi\varepsilon E\theta_1(\kappa\theta_2 IR)^{-1}\chi_0, \quad P = 2\pi\varepsilon E\theta_5\theta_1(\kappa H)^{-1}\chi_0,$$

where $I = H/R$, $\theta_5 = (v_2 + v_1 s)n_1 n_2((m_1 v_2^3 + m_2 v_1^3)E)^{-1}$.

When determining the stress state of the layer and foundation, most of the integrals can not be found analytically due to the complexity of the function $G(t)$. Therefore, starting from the second approximation of the function (1.17), we expand the expressions under integrals into power series h^{-i} $(i = 1, 2, \ldots)$; it allows us to calculate the coefficients of (1.18) approximately.

1.3.3 Numerical Results

Here, the impact of initial stresses on the contact interaction of the elastic cylinder and layer is studied for the potentials of Bartenev–Khazanovich, Treloar, and harmonic potential. We find a numerical solution of the quasi-regular system (1.18) at $k = 32$. The algorithm based on the reduction method was tested on some reference problems and showed its efficiency. Numerical analysis is presented in dimensionless coordinates (Figs. 1.2–1.6). In the figures, the dashed line corresponds to a stress state without initial stresses.

Figures 1.2 and 1.3 show the die pressure for the cylinder with initial stresses at $h = 1.6$. The impact of initial stresses on contact displacements in the cylinder and layer is illustrated in Figs. 1.4 and 1.5. Figure 1.6 shows that tangential stresses are the most concentrated near the contact zone.

By comparing the stress state of bodies with initial stresses and the corresponding state for an isotropic body without initial stresses, the following equation is obtained at $z_i = 0$

$$U_3(r,0) = kU_3^0(r,0), \quad Q_{33}(r,0) = k_s Q_{33}^0(r,0), \tag{1.19}$$

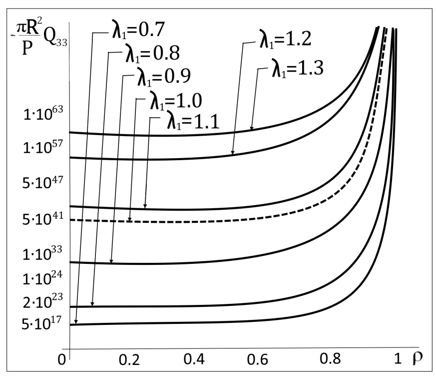

Figure 1.2 Harmonic potential.

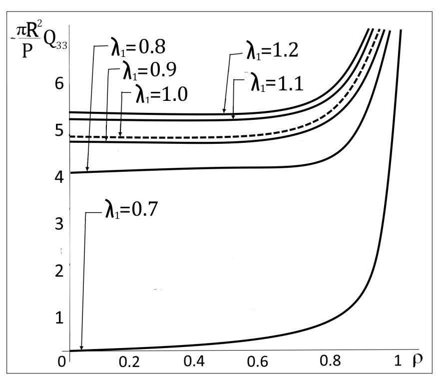

Figure 1.3 Treloar potential.

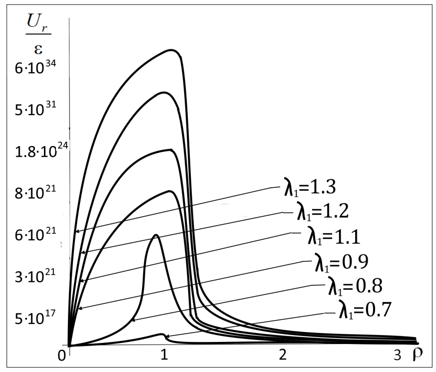

Figure 1.4 Bartenev–Khazanovich potential.

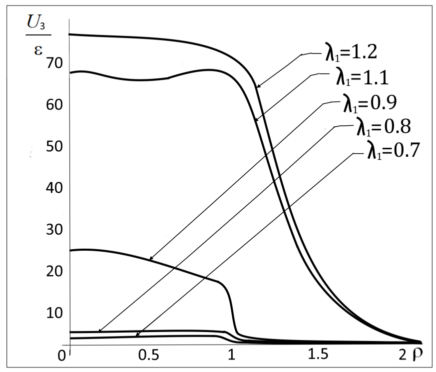

Figure 1.5 Treolar potential.

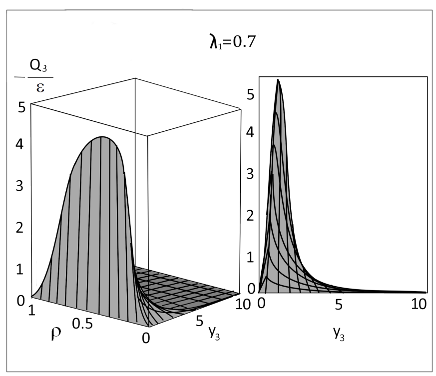

Figure 1.6 Treolar potential.

where $U_3(r,0)$ and $Q_{33}(r,0)$ are displacement and stress under the punch pressed into the layer with initial stresses; ${U_3}^0(r,0)$ and ${Q_{33}}^0(r,0)$ are displacement and stress under the punch pressed into the layer without initial stresses; k and k_s are coefficients that illustrate the effect of initial stresses on contact stresses and displacement of elastic cylinder and layer.

The dependence of coefficients changes k and k_s from (1.19) is presented in [16]. When the elongation coefficient approaches the value of the surface instability of the material, displacements increase unboundedly, and stresses tend to zero.

1.4 Pressure of Two Prestressed Half-Spaces on Elastic Cylindrical Punch With Initial (Residual) Stresses

The section presents the problem statement, boundary conditions, solution method, and numerical results for this problem.

1.4.1 Problem Statement and Boundary Conditions

Let the finite elastic cylindrical punch with initial stresses is compressed (stretched) by two identical prestressed half-spaces reduced to equal and oppositely directed forces P (Fig. 1.7). The geometrical axis of symmetry of the punch coincides with the y_3-axis of the cylindrical coordinate system (r, θ, y_3). The external load is applied so that the points of the unloaded surfaces of the bases move in the direction of $y_3 = 0$ by ε. Let $h = 0.5H$.

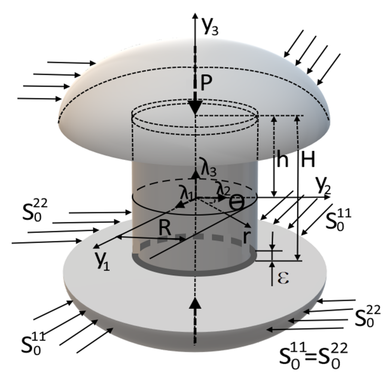

Figure 1.7 Two prestressed half-spaces and an elastic cylindrical punch with initial stresses.

The following boundary conditions correspond to the problem statement

1. at the base of the elastic punch

$$u_3'^{(i)} - u_3'^{(3)} = \varepsilon, \quad Q_{33}'^{(3)} = Q_{33}'^{(i)}, \quad Q_{3r}'^{(3)} = 0, \quad Q_{3r}'^{(i)} = 0$$
$$(z_i = \pm h/v_i, \quad i = 1, 2; \quad 0 \leqslant r \leqslant R), \quad (1.20)$$

2. on the boundaries of elastic half-spaces outside the contact area

$$Q'^{(3)}_{33} = 0, \quad Q'^{(i)}_{3r} = 0, \quad u'^{(i)}_{3} = 0, \quad (z_i = \pm h/v_i, \ \ i = 1, 2; \ \ r > R), \quad (1.21)$$

3. on the side surface of the elastic punch

$$Q'^{(3)}_{rr} = 0, \quad Q'^{(3)}_{3r} = 0 \quad (|z_i| \leqslant h/v_i, \ \ i = 1, 2; \ \ r = R). \quad (1.22)$$

The equilibrium condition that establishes the dependence of load P on displacements of punch bases is as follows:

$$P = -2\pi \int_0^R r \left| Q'^{(i)}_{33} \right| dr, \quad \left| Q'^{(i)}_{33} \right| = \left| Q'^{(i)}_{3r} \right|_{z_i = \pm H/v_i} \quad (i = 1, 2). \quad (1.23)$$

1.4.2 Method of Solution

In the case of unequal roots ($\xi'^2_2 \neq \xi'^2_3$) of the characteristic equation (1.4), the solution for a cylindrical elastic punch with initial stresses reads

$$\begin{aligned}
\tilde{\chi} &= \frac{\varepsilon \chi_0}{2h\theta_6} \left(\frac{r^2}{2}(z_1 + z_2) - \frac{1}{3}(z_1^3 + z_2^3) \right) \\
&+ \frac{\varepsilon \omega_2}{R n_1} \sum_{k=1}^{\infty} \mu_k \chi_k \left\{ \left[\frac{s_0 I_1(\gamma_k v_2 R)}{I_1(\gamma_k v_1 R)} I_0(\gamma_k v_1 r) \sin(\gamma_k v_1 z_1) - I_0(\gamma_k v_2 r) \sin(\gamma_k v_2 z_1) \right] \right. \\
&- J_0(\alpha_k r) F_k^* \left[\frac{v_2(\tilde{c}_0 - \tilde{c}_2) \left(\cot\left(\frac{\alpha_k h}{v_2}\right) \sinh\left(\frac{\alpha_k h}{v_1}\right) - \cosh\left(\frac{\alpha_k h}{v_1}\right) \right) \sinh(\alpha_k z_1)}{\sinh(\alpha_k h v_1^{-1}) \left(v_2(\tilde{c}_0 - \tilde{c}_2) + v_1 s_0 (\tilde{c}_1 - \tilde{c}_0) \right)} \right. \\
&+ \cosh(\alpha_k z_1) + \frac{n_2(\tilde{c}_1 - \tilde{c}_0) \left(\cot\left(\frac{\alpha_k h}{v_2}\right) \sinh\left(\frac{\alpha_k h}{v_1}\right) - \cosh\left(\frac{\alpha_k h}{v_2}\right) \right) \sinh(\alpha_k z_2)}{v_1 \sinh(\alpha_k h v_1^{-1}) \left(v_2(\tilde{c}_0 - \tilde{c}_2) + v_1 s_0 (\tilde{c}_1 - \tilde{c}_0) \right)} \\
&\left.\left. - \frac{n_2 \sinh(\alpha_k h v_1^{-1}) \cosh(\alpha_k z_2)}{n_1 s_0 \sinh(\alpha_k h v_2^{-1})} \right] \right\}, \quad (1.24)
\end{aligned}$$

where

$$\omega_2 = v_1^3 m_1^{-1} (s_3 - s_2)^{-1}, \ \ \gamma_k = \pi k h^{-1}, \ \ \alpha_k = \mu_k R^{-1} \ \ (J_1(\alpha_k R) = 0),$$

$$F_k^* = (1+\alpha_k^3)\gamma_k^3 \left[\frac{n_1 \tilde{A}_k^* G_k(1;R)}{1-\gamma_k^2 v_1^2} \left(\gamma_k v_1 \sin(\gamma_k h) \cos\left(\frac{h}{v_1}\right) - \cos(\gamma_k h) \sin\left(\frac{h}{v_1}\right)\right)\right.$$
$$\left. + \frac{n_2 G_k(2;R)}{1-\gamma_k^2 v_2^2} \left(\gamma_k v_2 \sin(\gamma_k h) \cos\left(\frac{h}{v_2}\right) - \cos(\gamma_k h) \sin\left(\frac{h}{v_2}\right)\right)\right]$$
$$\times \left(\alpha_k^3 J_0(\alpha_k R) \left[\frac{\tilde{c}_1 - \tilde{c}_0}{v_1} E_k^* \left(\alpha_k \sinh\left(\frac{\alpha_k h}{v_1}\right) \cos\left(\frac{h}{v_1}\right) + \cosh\left(\frac{\alpha_k h}{v_1}\right) \sin\left(\frac{h}{v_1}\right)\right)\right.\right.$$
$$\left.\left. + \frac{\tilde{c}_2 - \tilde{c}_0}{v_2} N_k^* \left(\alpha_k \sinh\left(\frac{\alpha_k h}{v_2}\right) \cos\left(\frac{h}{v_2}\right) + \cosh\left(\frac{\alpha_k h}{v_2}\right) \sin\left(\frac{h}{v_2}\right)\right)\right]\right).$$

The stress-strain state in prestressed half-spaces is defined by the following linearized equations [11]

$$\begin{aligned}
Q_{33}^{\prime(i)}(\rho;\zeta_i) &= \frac{C_{44}(1+m_1)l_1}{R} \int_0^\infty F(\eta)(s e^{\eta\zeta_2} - s_3 e^{\eta\zeta_1}) J_0(\eta\rho) d\eta, \\
Q_{3r}^{\prime(i)}(\rho;\zeta_i) &= -\frac{C_{44}(1+m_1)}{v_1 R} \int_0^\infty F(\eta)(e^{\eta\zeta_2} - e^{\eta\zeta_1}) J_1(\eta\rho) d\eta, \\
U_3^{\prime(i)}(\rho;\zeta_i) &= -\frac{m_1}{v_1} \int_0^\infty \frac{F(\eta)}{\eta}(s_2 e^{\eta\zeta_2} - s_3 e^{\eta\zeta_1}) J_0(\eta\rho) d\eta, \\
U_r^{\prime(i)}(\rho;\zeta_i) &= -\int_0^\infty \frac{F(\eta)}{\eta}(e^{\eta\zeta_2} - s_3 e^{\eta\zeta_1}) J_1(\eta\rho) d\eta,
\end{aligned} \tag{1.25}$$

where $\xi = z_i v_i R^{-1}$, $\zeta_i = \xi v_i^{-1} = z_i R^{-1}$, $\eta = \xi R$ $(i = 1,2)$, $s = s_0 l_2 l_1^{-1}$, $s_1 = (m_1 - 1)m_1^{-1}$, $s_2 = v_1 m_2 (v_2 m_1)^{-1}$, $s_3 = s_0 v_1 v_2^{-1}$ and $F(\eta)$ is the unknown function.

From the first boundary conditions in (1.20)–(1.21), the unknown function $F(\eta)$ can be defined from the dual integral equations

$$\begin{aligned}
&\int_0^\infty \frac{F(\eta)}{\eta} J_0(\eta\rho) d\eta = q(\rho) \quad (0 < \rho < 1), \\
&\int_0^\infty F(\eta) J_0(\eta\rho) d\eta = 0 \quad (\rho > 1),
\end{aligned} \tag{1.26}$$

where

$$q(\rho) = \varepsilon \left\{ (1-\chi_0)\frac{\omega_2}{n_1} + \sum_{k=1}^\infty \alpha_k J_0(\mu_k \rho) F_k^* \left[\frac{m_1}{n_1} \left(E_k^* \sinh\left(\frac{\mu_k h}{v_1 R}\right)\right.\right.\right.$$
$$\left.\left.\left. + \cosh\left(\frac{\mu_k h}{v_1 R}\right)\right) + \frac{m_2}{n_2}\left(N_k^* \sinh\left(\frac{\mu_k h}{R v_2}\right) + M_k^* \cosh\left(\frac{\mu_k h}{R v_2}\right)\right)\right] \chi_k \right\}.$$

Using the inversion formula [11], we obtain

$$\frac{F(\eta)}{\eta} = \frac{2\varepsilon}{\pi}\left((1-\chi_0)\frac{\omega_2}{n_1}\Psi(\eta,0) + \sum_{k=1}^{\infty}\frac{\mu_k}{R}F_k^*\left[\frac{m_1}{n_1}\left(E_k^*\sinh\left(\frac{\mu_k h}{v_1 R}\right)+\cosh\left(\frac{\mu_k h}{v_1 R}\right)\right) + \frac{m_2}{n_2}\left(N_k^*\sinh\left(\frac{\mu_k h}{R v_2}\right)+M_k^*\cosh\left(\frac{\mu_k h}{R v_2}\right)\right)\right]\chi_k\Psi(\eta,\mu_k)\right). \quad (1.27)$$

To determine the coefficients ${N_k}^*$, ${E_k}^*$, ${M_k}^*$ and the function $F(\eta)$, new variables are introduced:

$$\begin{gathered}\int_0^{\infty}\frac{F(\eta)}{\eta}J_1(\eta\rho)d\rho = -\frac{\varepsilon R(v_2+v_1 s)}{2v_1 v_2 h(s-s_3)\theta_6}\chi_0, \quad \chi_k = -\frac{Rn_1}{\varepsilon\mu_k\omega_2}\tilde{B}_k, \\ \int_0^{\infty}\eta\Psi(\eta,\mu_k)\int_0^1\rho J_0(\mu_n\rho)J_0(\eta\rho)d\rho d\eta = \Psi(\mu_n,\mu_k).\end{gathered} \quad (1.28)$$

The infinite system, which is similar to (1.18) for the unknowns χ_k $(k = 0, 1, \dots)$ in (1.24)–(1.27), is obtained. Thus, this problem is also reduced to the determination of constants χ_k $(k = 0, 1, 2, \dots)$ that determine the stress state of the elastic punch and two half-spaces with initial stresses. Using the equilibrium condition (1.23), the formula for the load P is found as $P = \pi\varepsilon R^2 C_{44}(1+m_1)l_1(v_2+sv_1)(v_2v_1h\theta_6)^{-1}$.

After defining the unknowns χ_k $(k = 0, 1, 2, \dots)$ from a system of linear algebraic equations similar to (1.18), it is possible to obtain displacements and stresses in both elastic half-spaces and elastic punch [18]. Problem solutions are also represented as series containing an infinite system of constants χ_k $(k = 0, 1, \dots)$.

1.4.3 Numerical Results

Numerical analysis is presented for the Treloar potential (for Neo-Hookean bodies). The distribution of normal stresses ${Q_{33}}'^{(3)}/P$ in the contact zone (at $z_i = h/v_i$) and along the elastic cylinder is shown in Fig. 1.7. Displacement ${U_3}'^{(3)}/\varepsilon$ in the contact zone (at $z_i = h/v_i$) and along the cross-section of the elastic cylinder (at $z_i = 0$) is shown in Fig. 1.8, where the dashed curve corresponds to contact without initial stresses ($\lambda_1 = 1$), and the solid curve – with initial stresses. All quantities in Figs. 1.9 and 1.10 are presented in dimensionless coordinates. When initial (residual) stresses are absent ($\lambda_1 = 1$), the graph of contact stresses distribution corresponds to the known solutions of the contact problem about the pressure of two half-spaces on a cylinder without accounting for initial stresses [1]. The dependence of the equivalent load P on the elongation coefficient λ_1 is illustrated in Fig. 1.10 for some argument values.

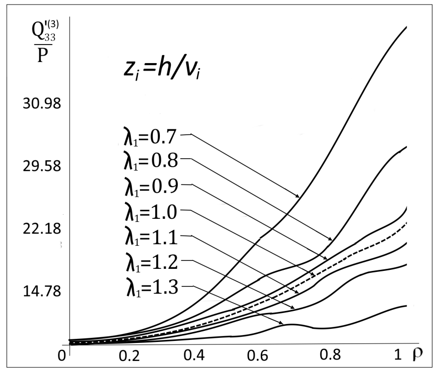

Figure 1.8 Contact stresses Q_{33}/P.

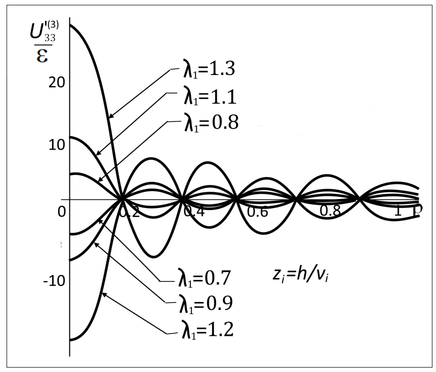

Figure 1.9 Contact displacement $U_3^{'(3)}/\varepsilon$ in the contact zone.

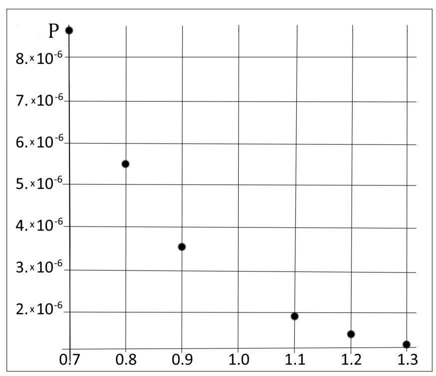

Figure 1.10 Dependence of equivalent force P on the elongation coefficient λ_1.

1.5 Contact Problem for a Rigid Ring Punch With Half Space With Initial (Residual) Stresses

The section presents the problem formulation, boundary conditions, solution method, and numerical results for the problem.

1.5.1 Problem Statement and Boundary Conditions

Let a finite rigid ring-shaped punch (with a flat base) is pressed into a prestressed half-space with force P (Fig. 1.11). The geometric symmetry axis of the punch coincides with the y_3-axis (in the cylindrical coordinate system) directed into the half-space. The geometry parameters R_1 and R_2 are the inner and outer radii of the punch, respectively. The external load is assumed to be applied only to the free end of the elastic punch. All the points of the punch base move to the direction of the symmetry axis y_3 by the same value ε.

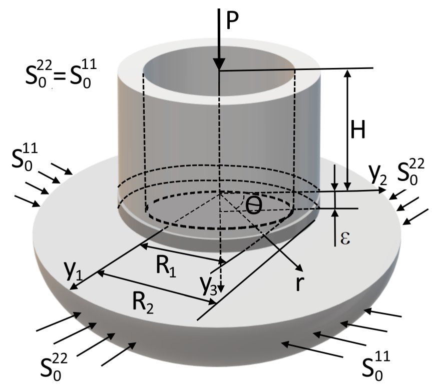

Figure 1.11 Rigid ring-shaped punch on elastic half-space with initial stresses.

In the circular cylindrical coordinate system (r, θ, z_i), the following boundary conditions correspond to the given above problem statement :

$$U_z = \varepsilon \ (R_1 < r < R_2), \tag{1.29}$$
$$Q_{zz} = 0 \ (0 < r < R_1 \ R_2 < r < \infty), \tag{1.30}$$
$$Q_{rz} = 0 \ (0 < r < \infty), \tag{1.31}$$
$$U_z = 0 \ (0 < r < R_1, \ R_2 < r < \infty), \tag{1.32}$$
$$Q_{zz} = -\sigma_z^0 \ (R_1 < r < R_2), \tag{1.33}$$

where $\sigma_z^0 = 8\varepsilon\omega_3(\pi\sqrt{1-r^2})^{-1}$ for $R_1 < r < R_2$, and $\sigma_z^0 = 0$ for $0 < r < R_1$, $R_2 < r < \infty$; $\omega_3 = C_{44}(1+m_1)l_1(s-s_0)$.

The equilibrium condition that establishes the relation between the punch indentation and the equivalent load P is as follows

$$P = -2\pi \int_{R_1}^{R_2} rQ_{zz}(0, r)dr. \tag{1.34}$$

1.5.2 Method of Solution

The stress-strain state in prestressed half-space is determined based on conditions (1.29)–(1.33) and $z_1 = 0$ for unequal roots by the following quantities [14, 20]

$$\begin{aligned} Q_{33} &= \frac{\omega_4}{R_0}\int_0^\infty F(\eta)J_0(\eta r)d\eta, \\ U_3 &= -\frac{1}{\omega_5}\int_0^\infty \frac{F(\eta)}{\eta}J_0(\eta r)d\eta, \\ U_r &= \omega_6\int_0^\infty \frac{F(\eta)}{\eta}J_1(\eta r)d\eta, \end{aligned} \tag{1.35}$$

where

$$R_0 = \frac{R_1}{R_2}, \quad \omega_4 = C_{44}(1+m_1)l_1(s-s_3), \quad \omega_5 = \frac{v_1}{m_1(s_2-s_3)}, \quad \omega_6 = s_3 - 1.$$

Taking into account condition (1.33), we obtain:

$$\frac{\omega_4}{R_0}\int_0^\infty F(\eta)J_0(\eta r)d\eta = 0 \quad (0 < r < R_1, \;\; R_2 < r < \infty) \tag{1.36}$$

Introducing the unknown continuous function $f(r)$ $(R_1 < r < R_2)$ defining the distribution of die pressure, and extending (1.36) to $0 < r < \infty$, we get [20]

$$\frac{\omega_4}{R_0}\int_0^\infty F(\eta)J_0(\eta r)d\eta = f(r)\left(\delta(r-R_1) - \delta(r-R_2)\right) \quad (0 < r < \infty) \tag{1.37}$$

where $\delta(r)$ is the delta function.

Since the function $f(r)$ ($f(r) = 0$ at $r \leqslant R_1$ and $r \geqslant R_2$) is continuous, we present it by a segment of the generalized Furrier series [14, 20]:

$$L_n(r) = J_0\left(\frac{\gamma_n}{R_1}r\right)Y_0(\gamma_n) - J_0(\gamma_n)Y_0\left(\frac{\gamma_n}{R_1}r\right),$$

where γ_n are the positive roots of the equation

$$J_0\left(\frac{\gamma_n}{R_1}R_2\right)Y_0(\gamma_n) - J_0(\gamma_n)Y_0\left(\frac{R_2}{R_1}\gamma_n\right) = 0,$$

$Y_0(x)$ is the Weber function. Thus,

$$f(r) = \sum_{n=1}^{\infty} a_n L_n(r),$$

where a_n are unknown coefficients.

Applying the inversion formula of the integral Hankel transform to (1.37), we get

$$\frac{F(\eta)}{\eta} = \frac{R_0}{\omega_4} \sum_{n=1}^{\infty} a_n \Phi_n(\eta) \quad (0 < \eta < \infty), \tag{1.38}$$

where

$$\Phi_n(\eta) = \int_{R_1}^{R_2} r L_n(r) J_0(\eta r) dr$$
$$= \frac{\gamma_n \eta^2}{\gamma_n^2 - (\eta R_1)^2} \left\{ \frac{R_2}{R_1} \left[J_1\left(\frac{R_2}{R_1}\gamma_n\right) Y_0(\gamma_n) - Y_1\left(\frac{R_2}{R_1}\gamma_n\right) J_0(\gamma_n) \right] \right.$$
$$\left. \times J_0(\eta R_2) - \left[J_1(\gamma_n) Y_0(\gamma_n) - Y_1(\gamma_n) J_0(\gamma_n) \right] J_0(\eta R_1) \right\}.$$

Using the second expression in (1.35), equation (1.38) and (1.40), we obtain

$$\sum_{n=1}^{\infty} a_n \phi_n(r) = \frac{\varepsilon}{k_1} \quad (R_1 < r < R_2), \tag{1.39}$$

where

$$k_1 = -\frac{R_0}{\omega_4 \omega_5}, \quad \phi_n(r) = \int_0^{\infty} \Phi_n(\eta) J_0(\eta r) d\eta.$$

After determining a_n from (1.39), it is possible to obtain the components of the stress-strain state in the elastic half-space using (1.38) and (1.35). The relation between the indentation displacement and the equivalent force P, according to (1.34), reads $P = 16\omega_3 \varepsilon (1 - \sqrt{1 - R_0})$.

1.5.3 Numerical Results

This section presents a numerical analysis corresponding to the Treloar potential (Neo-Hookean bodies). The normal contact stress $\varepsilon^{-1} Q_{33}$ and radial displacement $\varepsilon^{-1} U_r$ is shown in Figs. 1.12 and 1.13 in dimensionless coordinates. The larger the elongation coefficients λ_1, the higher the curve for mentioned relations. Dotted curves correspond to half-space without initial stresses ($\lambda_1 = 1$), and solid curves – with initial stresses.

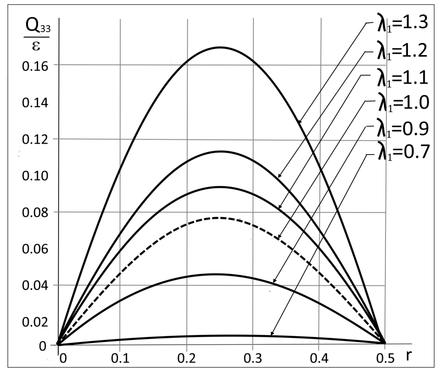

Figure 1.12 Contact stresses $Q_{33}\,\varepsilon^{-1}$. The potential of Treloar.

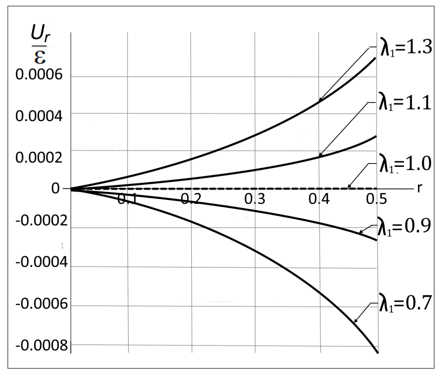

Figure 1.13 Contact displacement $U_r\,\varepsilon^{-1}$. The potential of Treloar.

1.6 Pressure of Elastic Ring Punch With Initial Stresses on Prestressed Half-Space

The section presents the formulation of the problem, boundary conditions, solution method, and numerical results of the problem on the indentation of an elastic ring-shaped punch with initial stresses into a prestressed half-space.

1.6.1 Problem Statement and Boundary Conditions

Let a finite elastic ring-shaped punch of height H is pressed into a half-space with force P. The geometric axis of symmetry coincides with the axis y_3 of the cylindrical coordinate system (r, θ, y_3) and is directed inside the half-space (Fig. 1.14). Geometry parameters R_1 and R_2 are the inner and outer radii of the punch, respectively. The external load is assumed to be applied only to the free end of the punch, where all points move along the axis of symmetry y_3 by the same value ε. The punch is assumed to be prestressed.

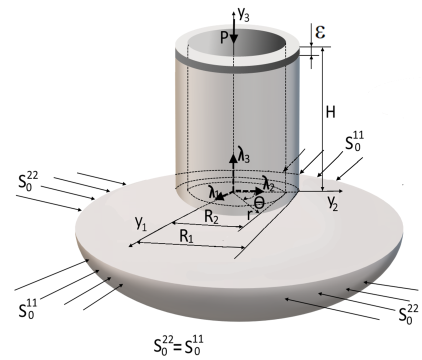

Figure 1.14 Elastic ring-shaped punch and elastic half-space with initial stresses.

In the system of cylindrical coordinates (r, θ, y_3), the following equations correspond to the boundary conditions:

$$U_3'^{(1)} = -\varepsilon, \quad Q_{3r}'^{(1)} = 0 \quad (z_i = Hv_i^{-1}, \ i = 1, 2; \ R_1 < r < R_2), \tag{1.40}$$

$$U_3'^{(1)} = U_3'^{(2)}, \quad Q_{33}'^{(1)} = Q_{33}'^{(2)}, \quad Q_{3r}'^{(1)} = Q_{3r}'^{(2)} = 0 \quad (z_i = 0, \ i = 1, 2; \ R_1 < r < R_2), \tag{1.41}$$

$$Q_{33}'^{(2)} = 0, \quad Q_{3r}'^{(2)} = 0 \quad (z_i = 0, \ i = 1, 2; \ 0 < r < R_1, \ R_2 < r < \infty), \tag{1.42}$$

$$Q_{rr}'^{(1)} = 0, \quad Q_{3r}'^{(1)} = 0 \quad (0 \leqslant z_i \leqslant Hv_i^{-1}, \ r = R_1, \ r = R_2). \tag{1.43}$$

The equilibrium condition that establishes the relation between the indentation displacement and the equivalent force P is as follows

$$P = -2\pi \int_{R_1}^{R_2} r Q_{33}'^{(2)}(0, r) dr. \tag{1.44}$$

1.6.2 Method of Solution

To determine the stress-strain state in an elastic ring-shaped punch with initial stresses, was use the linearized equations (see p. 78 in [15]). These equations give the displacement and the stress for compressible and non-compressible bodies. The general solution $\chi = \chi_1 + \chi_2$ for the case of unequal roots of the characteristic equation [15] is obtained in the form

$$\tilde{\chi} = 2A_0(r^2 - z_1^2 - z_2^2) + 3r^2 C_0(z_1^3 + z_2^3)$$
$$+ \sum_{k=1}^{\infty} \Big\langle C_k \Big\{ \Big[A_k^{(1)} I_0(\gamma_k v_1 r) + A_k^{(2)} K_0(\gamma_k v_1 r) \Big] \sin(\gamma_k v_1 z_1)$$
$$+ \Big[A_k^{(1)} I_0(\gamma_k v_1 r) + A_k^{(2)} K_0(\gamma_k v_1 r) \Big] \sin(\gamma_k v_1 z_1) \Big\}$$
$$+ M_k \Big[T_k^{(1)} J_0(\alpha_k r) + T_k^{(2)} Y_0(\alpha_k r) \Big] \big(\tilde{S}_2(\alpha_k z_1) + \tilde{S}_3(\alpha_k z_2) \big) \Big\rangle,$$

where $A_k^{(2)}$, $B_k^{(2)}$, $T_k^{(2)}$, C_0, C_k, $\tilde{M}_k$ = const, and M_k are the unknowns,

$$\begin{aligned}
&\tilde{S}(x) = \tilde{E}_k \sinh(x) + \tilde{F}_k \cosh(x), \quad \tilde{S}_3(x) = \tilde{N}_k \sinh(x) + \cosh(x),\\
&T_k^{(1)} = -Y_1(\alpha_k R_1 R_2^{-1})(J_1(\alpha_k R_1 R_2^{-1}))^{-1} T_k^{(2)},\\
&A_k^{(1)} = K_1(\gamma_k v_1 R_1)(I_1(\gamma_k v_1 R_1))^{-1} A_k^{(2)},\\
&B_k^{(1)} = K_1(\gamma_k v_2 R_1)(I_1(\gamma_k v_2 R_1))^{-1} B_k^{(2)},\\
&\tilde{N}_k = -\coth(\alpha_k H v_2^{-1}), \quad \tilde{S}_4(x) = \tilde{E}_k \cosh(x) + \tilde{F}_k \sinh(x),\\
&A_0 = (3C_0 H(v_1)^{-1} - \varepsilon n_1 n_2 (4(m_1 n_2 + m_2 n_1))^{-1}),\\
&M_k = \tilde{M}_k T_k^{(2)}, \quad \tilde{E}_k = (1 + m_2) n_1 ((1 + m_1) n_2)^{-1} \coth(\alpha_k H v_1^{-1}),\\
&\tilde{F}_k = -(1 + m_2) n_1 ((1 + m_1) n_2)^{-1}, \quad \gamma_k = \pi k H^{-1},\\
&\alpha_k = \mu_k R_2 R_1^{-1}, \quad J_1(\mu_k) Y_1(\mu_k R_2 R_1^{-1}) - Y_1(\mu_k) J_1(\mu_k R_2 R_1^{-1}) = 0.
\end{aligned}$$

Basing on (1.40)–(1.43) and $z_1 = 0$, the stress-strain state in the prestressed half-space is obtained for unequal roots in the form [10, 11, 15]:

$$\begin{aligned}
Q_{33}^{(2)} &= \frac{\omega_3}{R_2 - R_1} \int_0^{\infty} F(\eta) J_0(\eta r) d\eta,\\
U_3^{(2)} &= -\frac{1}{\omega_2} \int_0^{\infty} \frac{F(\eta)}{\eta} J_0(\eta r) d\eta,\\
U_r^{(2)} &= \omega_1 \int_0^{\infty} \frac{F(\eta)}{\eta} J_1(\eta r) d\eta,
\end{aligned} \qquad (1.45)$$

where $\omega_2 = v_1 \left(m_1 (s_3 - s_2) \right)^{-1}$, $\omega_1 = s_0 - 1$.

Satisfying the first condition (1.41), we define the unknown function $F(\eta)$ in (1.45) from of the triple integral equations

$$\begin{aligned}
&\int_0^{\infty} F(\eta) J_0(\eta r) d\eta = 0 \qquad (R_2 < r < \infty),\\
&\int_0^{\infty} \frac{F(\eta)}{\eta} J_0(\eta r) d\eta = f(r) \ (R_1 < r < R_2),\\
&\int_0^{\infty} F(\eta) J_0(\eta r) d\eta = 0 \qquad (0 < r < R_1),
\end{aligned} \qquad (1.46)$$

where

$$f(r) = \frac{\omega_2}{R_2}\left(\varepsilon + t_1 \sum_{k=1}^{\infty} \alpha_k^2 \left(\frac{Y_1(\alpha_k R_1 R_2^{-1})}{J_1(\alpha_k R_1 R_2^{-1})} J_0(\alpha_k R_2^{-1} r) - Y_0\big(\alpha_k R_2^{-1} r\big)\right) M_k\right),$$
$$t_1 = \frac{m_1 - m_2}{n_2(1+m_1)}.$$

Further, we reduce the integral equations (1.46) to a single equation [21]. The function $F(\eta)$ then can be sought in the form [21]:

$$F(\eta) = R_2 \sum_{n=0}^{\infty} W_{2n} J_{2n}\left(\tfrac{1}{2}5\eta(R_2 - R_1)\right) J_{2n}\left(\tfrac{1}{2}\eta(R_2 + R_1)\right), \tag{1.47}$$

where W_{2n} are unknown constants. Substituting (1.47) in (1.46), we get the integral equation

$$\sum_{n=0}^{\infty} W_{2n} \int_0^{\infty} J_{2n}\left(\tfrac{1}{2}\eta(R_2 - R_1)\right) J_{2n}\left(\tfrac{1}{2}\eta(R_2 + R_1)\right) J_0(\eta r) d\eta = f(r). \tag{1.48}$$

From the second boundary condition (1.41), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} W_{2n} \int_0^{\infty} & J_{2n}\left(\tfrac{1}{2}\eta(R_2 - R_1)\right) \\ & \times J_{2n}\left(\tfrac{1}{2}\eta(R_2 + R_1)\right) \int_{R_1}^{R_2} r J_0(\mu_k r) J_0(\eta r) dr d\eta \\ & = \frac{\hat{N}_{44}(R_2 - m_1)(1+m_2)}{\omega_3 v_2} \alpha_k^2 t_1 \left[\frac{Y_1(\alpha_k R_1 R_2^{-1})}{J_1(\alpha_k R_1 R_2^{-1})} \tilde{O}_k^{(1)} - \tilde{O}_k^{(2)}\right] M_k. \end{aligned} \tag{1.49}$$

where

$$\tilde{O}_k^{(1)} = \frac{R_2}{\alpha_k}\left[R_1 J_1\big(\alpha_k R_1 R_2^{-1}\big) - R_2 J_1(\alpha_k)\right],$$
$$\tilde{O}_k^{(2)} = \frac{R_2}{\alpha_k}\left[R_1 Y_1\big(\alpha_k R_1 R_2^{-1}\big) - R_2 Y_1(\alpha_k)\right].$$

To determine the constants M_i and W_{2i} $(i = 0, 1, \dots)$ from (1.45) and (1.46), we use an infinite system of linear algebraic equations consisting of (1.48) and (1.49). This system is solved by the reduction method, taking into account that $W_0 = \omega_2 \varepsilon \pi (R_2 - R_1)(8\omega_3 R_2)^{-1}$. Using the equilibrium condition (1.44), we establish the relation between the indention displacement and the equivalent force P in the following form

$$P = \frac{2\omega_2\omega_3\varepsilon}{\pi(R_2 - R_1)}.$$

Having determined the unknown constants M_i and W_{2i} $(i = 0, 1, \dots)$, we can obtain displacement and stress in the elastic half-space as well as in the punch. It

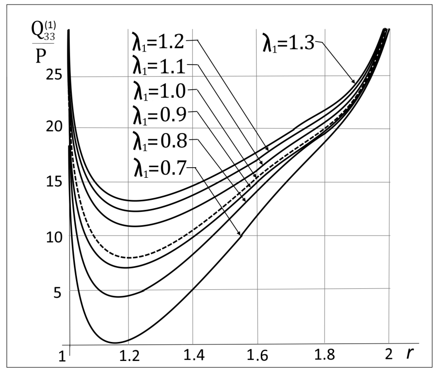

Figure 1.15 Contact stress.

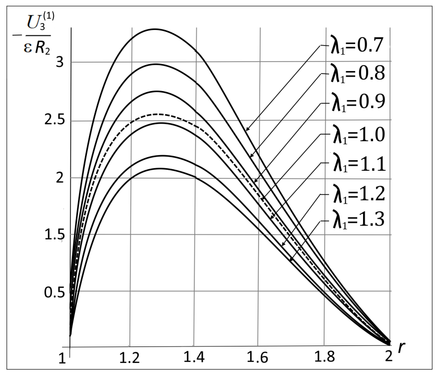

Figure 1.16 Contact displacement.

should be noted that the coefficients depend on the elastic potential and the height of the elastic punch H.

1.6.3 Numerical Results

In this section, numerical solutions corresponding to the Treloar potential are presented. Figures 1.15 and 1.16 show the normal contact stress $P^{-1}Q_{33}{}^{(1)}$ distribution and displacement $-(\varepsilon R_2)^{-1}U_3{}^{(1)}$ under the ring-shaped punch at the contact area boundary in dimensionless coordinates. The dashed curve corresponds to the half-space without initial stresses ($\lambda_1 = 1$), and the solid curve – with initial stresses.

1.7 Conclusion

The general engineering conclusions for the considered problems are as follows:

1. initial stresses in bodies lead to a decrease in stresses and an increase in the absolute values of displacements in compression ($\lambda_1 < 1$) an increase in stresses and a decrease in the absolute values of displacements in tension ($\lambda_1 > 1$). Thus, a prestressed state during the contact interaction of elastic bodies makes it possible to adjust contact stress and displacement when determining the strength of structures and their members. Moreover, for contact stresses, initial stresses are dangerous in case of tension, and for displacements, initial stresses are dangerous in case of compression;
2. more significantly, in quantitative terms, initial stresses act in highly elastic materials in comparison with more rigid ones, but their influence remains qualitatively;
3. in cases of absence of initial stresses, the obtained results coincide with the classical ones.

In the case of the pressure of two prestressed half-spaces on the elastic cylindrical punch and the pressure of the cylindrical punch on the layer, it was established that the most significant effect of initial stresses is observed on the side surface of the punches. The thickness of the layer does not affect the nature of the initial stress and affects only their value. The situation is dangerous when initial stresses approach the values of surface instability since contact stresses and displacement dramatically change their values.

Thus, from the study of the contact interaction of two prestressed half-spaces and an elastic cylindrical punch, it can be concluded that the closer to the central cross-section of the cylindrical punch, the faster the normal stresses tend to zero. Furthermore, displacement ${U_3}'^{(3)}/\varepsilon$ takes significantly higher values closer to the axis of the cylindrical punch than to its side surface. The values of the equivalent force P decrease with an increase in the elongation coefficient λ_1. Thus, the force P takes greater values during tension than during compression.

The research made it possible to

1. obtain analytical and numerical dependencies that determine the behavior of stresses and displacements in the contact zone;
2. develop algorithms for numerical determination of contact characteristics;
3. use the proposed research principle and solution algorithm when designing technological types of equipment, buildings, and other structures.

Thus, this chapter presents research on contact problems for prestressed bodies with elastic or rigid punches. We believe that further progress in the development of the mechanics of contact interaction of prestressed bodies (both in the case of rigid and elastic punches) is determined by the research of more complex classes of problems, for example, taking into account friction, conducting experimental researches on the effect of initial stresses on the main characteristics of contact interaction of structural materials.

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